University of Illinois

Second Mid-Semester Exam
Page 1 of 3

ECE 361
Spring 1998

Instructions:

There are **four problems** on this examination.

One page of notes allowed. No other notes, books, tables of integrals, and calculators/personal computers permitted.

Show all your work in the exam booklet provided. Answers without appropriate justification will receive no credit.

Notation: For $\lambda > 0$, $p_\lambda(t) = \begin{cases} 1 & \text{if } 0 \le t < \lambda, \\ 0 & \text{otherwise.} \end{cases}$

$\Phi(x)$ = cumulative probability distribution function for standard (zero-mean unit-variance) Gaussian random variable

$Q(x) = 1 - \Phi(x)$

$P_{e,i}$ = probability of error given that signal $s_i(t)$ was transmitted

AWGN denotes additive (zero-mean) white Gaussian noise with power spectral density $N_0/2$. This random process is independent of the choice of transmitted signal.

Some more-or-less useful facts:

$2\bullet\cos^2(x) = 1 + \cos(2x)$ $\qquad$ $2\bullet\sin^2(x) = 1 - \cos(2x)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ $\qquad$ $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\text{rect}(t/T) \leftrightarrow T\bullet\text{sinc}(fT)$ and $W\bullet\text{sinc}(Wt) \leftrightarrow \text{rect}(f/W)$ where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

$\cos(2\pi f_c t) \leftrightarrow [\delta(f+f_c) + \delta(f-f_c)]/2$ $\qquad$ $\sin(2\pi f_c t) \leftrightarrow j[\delta(f+f_c) - \delta(f-f_c)]/2$

Rayleigh density function: $\quad f(r) = \frac{r}{\sigma^2} \bullet \exp\left(\frac{-r^2}{2\sigma^2}\right),\ r > 0$

Rician density function: $\quad f(r) = \frac{r}{\sigma^2} \bullet I_0\left(\frac{rA}{\sigma^2}\right) \bullet \exp\left(\frac{-r^2-A^2}{2\sigma^2}\right),\ r > 0,$

$$\text{where } I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x\bullet\cos\theta)\ d\theta$$

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1. **(35 points)** Let $\{b_k\}$ denote a sequence of independent data bits each likely to take on values 0 and 1 with equal probability. This sequence is to be transmitted over an AWGN channel at the (slow!) data rate of 0.5 bit/second using energy E per bit.

Consider first a binary communication system using the signal

$$\sum_{k=-\infty}^{\infty} \sqrt{E}\bullet p_1(t - 2k - b_k)$$

(a) Sketch the four possible signals that can be observed in the time interval [0, 4).

Consider next a 4–ary communication system using the signal

$$\sum_{k=-\infty}^{\infty} \sqrt{2E}\bullet p_1(t - 4k - b_{2k} - 2b_{2k+1})$$

(b) Sketch the four possible signals that can be observed in the time interval [0, 4).

(c) **State** the bit error probabilities of these two communication systems. Which has larger bit error probability?

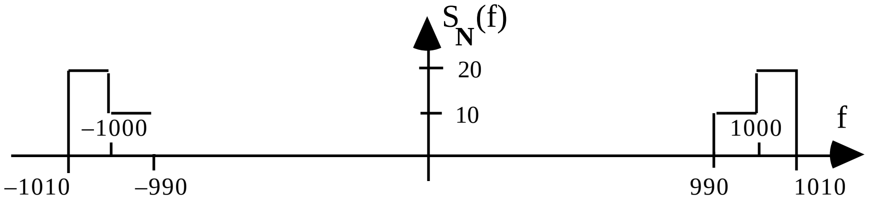
(d) Compare the bandwidth required to implement these communication systems. Which has a larger bandwidth?

(e) Write an expression similar to the ones above for the transmitted signal in a communication system using 8-ary orthogonal signaling with (positive) rectangular pulses. The data rate must be 1/2 bit/second and the energy per bit must be E (just as with the two systems discussed above.) What is the bandwidth required for this signal set? What is the peak power requirement compared to the binary and 4-ary systems described above?

2. **(35 points)** A zero-mean wide-sense-stationary narrowband random process $\{\mathbf{N}(t)\}$ has power spectral density $S_{\mathbf{N}}(f)$ as shown below (not to scale). The quadrature components of $\{\mathbf{N}(t)\}$with respect to the center frequency $f_c = 1$ kHz are given by

$$\mathbf{N}(t) = \mathbf{X}(t)\cdot\cos(2\pi(1000)t) - \mathbf{Y}(t)\cdot\sin(2\pi(1000)t)$$

where$\{\mathbf{X}(t)\}$ and $\{\mathbf{Y}(t)\}$ are low-pass processes.



(a) Find the autocorrelation function $R_{\mathbf{N}}(t)$ of the process by computing the inverse Fourier transform of $S_{\mathbf{N}}(f)$.

(b) Find the power spectral density $S_{\mathbf{X}}(f)$ of the random process $\{\mathbf{X}(t)\}$.

(c) Find the autocorrelation function $R_{\mathbf{X}}(t)$ of the random process $\{\mathbf{X}(t)\}$by computing the inverse Fourier transform of $S_{\mathbf{X}}(f)$.

(d) Find the autocorrelation function $R_{\mathbf{X}}(t)$ of the random process $\{\mathbf{X}(t)\}$ via the integral formula $R_{\mathbf{X}}(t) = 2\int_0^\infty S_{\mathbf{N}}(f)\cdot\cos(2\pi(f-1000)t)df$ and compare to your result in part (c).

3. **(30 points)** For $0 \le t \le 2T$, a QPSK signal is given by

$$\sqrt{E/T}\cdot[(-1)^{a_0}\cdot\cos(2\pi f_c t) - (-1)^{a_1}\cdot\sin(2\pi f_c t)].$$

An optimum coherent receiver for this signal (in an AWGN channel) uses local reference carriers $2\cdot\cos(2\pi f_c t)$ and $-2\cdot\sin((2\pi f_c t)$ that are derived from a single VCO source via phase delays (as described in class and in homework.)

(a) **State** the bit error probabilities in the I and Q branches of the optimum receiver in terms of one or more of the parameters E, T, f_c, and N_0. You are **not asked** to provide a detailed analysis of how you obtained this result.

(b) Now suppose that due to phase-tracking errors in the PLL and component imperfections in the phase-delay circuitry, the local reference carriers are actually $2\cdot\cos(2\pi f_c t+\theta)$ and $-2\cdot\sin((2\pi f_c t+\psi)$ where θ and ψ are small compared to $\pi/2$.

What are the error probabilities in the I and Q branches if $a_0 = 0$ and $a_1 = 1$?

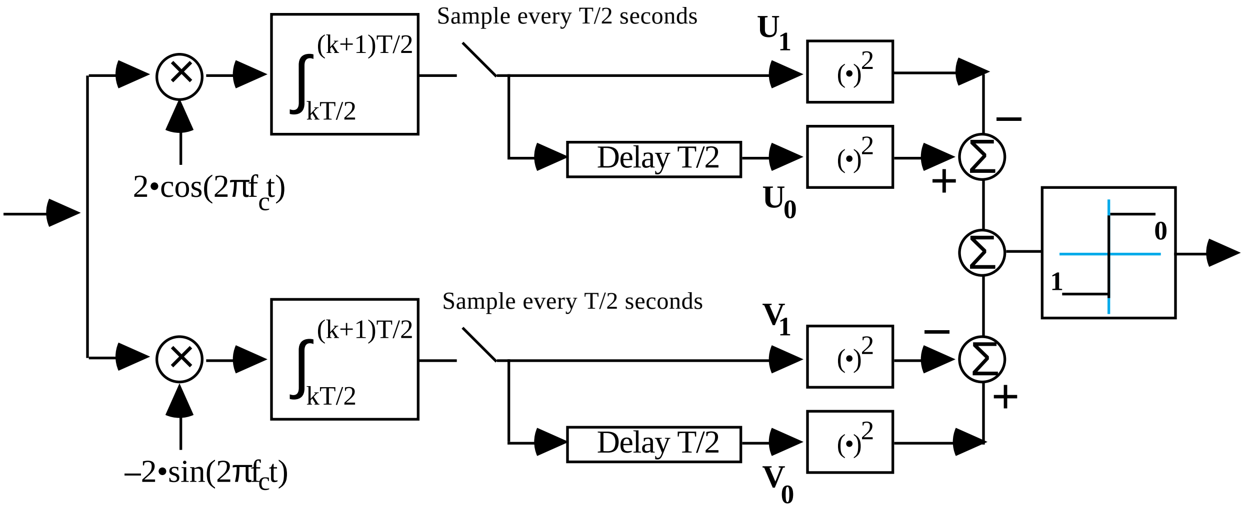
4. **(25 points)** A PPM system operating on an AWGN channel uses signals

$$s_0(t) = \sqrt{2}A \cdot p_{T/2}(t) \cdot \cos(2\pi f_c t+\theta)$$

$$s_1(t) = \sqrt{2}A \cdot p_{T/2}(t-T/2) \cdot \cos(2\pi f_c t+\theta)$$

where f_cT is a large even integer and θ is unknown.

A noncoherent correlation receiver for this signal set is shown below where the integrator outputs are sampled at integer multiples of T/2. The diagram indicates the state of affairs at time T. **Note that the integrator is dumped (that is, its output is reset to 0 and the integration is restarted) at the end of each T/2 second period. The decision is made once every T seconds.**



As shown, $\mathbf{U}_0$ and $\mathbf{V}_0$ denote the sample values at time t = T/2 and $\mathbf{U}_1$ and $\mathbf{V}_1$ denote the sample values at time t = T. The receiver computes $\{(\mathbf{U}_0)^2 + (\mathbf{V}_0)^2\} - \{(\mathbf{U}_1)^2 + (\mathbf{V}_1)^2\}$ and decides that $s_0(t)$ was transmitted if this quantity is positive.

(a) Explain why $s_0(t)$ and $s_1(t)$ are orthogonal over the interval [0,T] even though they are at the same carrier frequency f_c and have the same phase θ.

(b) Compute the means and variances of $\mathbf{U}_0$, $\mathbf{V}_0$, $\mathbf{U}_1$, and $\mathbf{V}_1$ conditioned on $s_0(t)$ being transmitted and $\theta = \pi/4$.

(c) **Use** the results of part (b) to **state** the error probability of this receiver in terms of one or more of the parameters A, T, f_c, θ and N_0. You are **not asked** to provide a derivation of your answer via integration of pdfs etc.

University of Illinois

Solutions to Second Mid-Semester Exam

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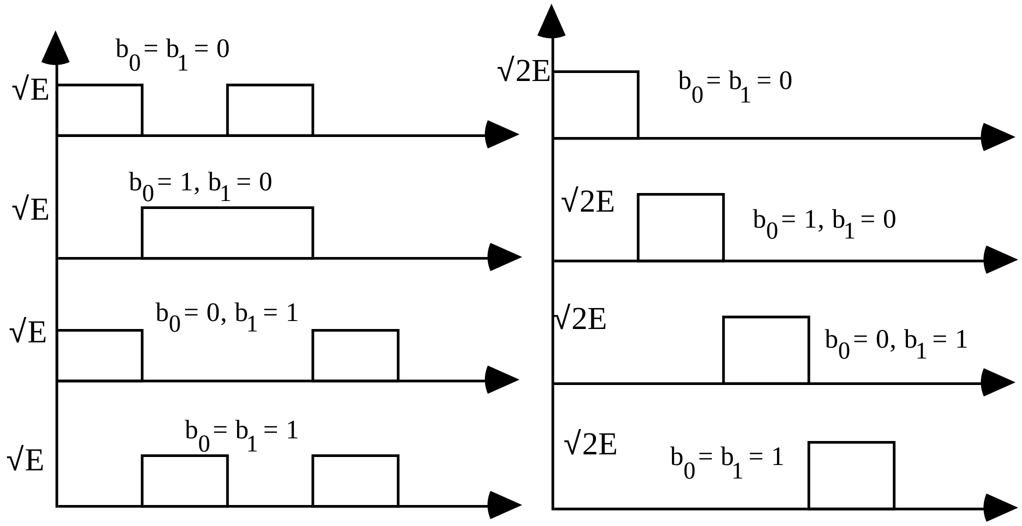
1.(a), (b) The four possible signals are as shown below.



(c) The first system uses *binary orthogonal signaling* with energy E per bit and hence its bit error probability is $Q(\sqrt{E/N_0})$. The second system uses *4-ary orthogonal signaling* with symbol energy 2E per symbol, and hence, according to a formula discussed in class, its bit error probability is given by

$$\left[\frac{4}{2\cdot(4-1)}\right]\cdot P_{e,s} = (2/3)\cdot P_{e,s} = (2/3)\cdot\int_{-\infty}^{\infty} 3\cdot[\Phi(x)]^2\cdot\varphi(x)\cdot\Phi\left(x-\sqrt{4E/N_0}\right)dx \le 2\cdot\int_{-\infty}^{\infty}\varphi(x)\cdot\Phi\left(x-\sqrt{4E/N_0}\right)dx$$

$= Q(\sqrt{2E/N_0})$ where we got the upper bound by replacing $\Phi(x)$ with 1 and then interpreted the integral as $P\{\mathbf{X} < \mathbf{Y}\}$ where $\mathbf{X}$ and $\mathbf{Y}$ are independent $N(\sqrt{4E/N_0},1)$ and $N(0,1)$ random variables. Since $Q(\sqrt{2E/N_0}) << Q(\sqrt{E/N_0})$, we can be sure that the first system has larger bit error probability for large E/N_0. However, for *very small* E/N_0, binary orthogonal signaling *can* achieve a smaller bit error rate.

(d) From the figure above, it seems reasonable to assume that both systems use the same bandwidth. Note that the Landau-Pollak theorem states that there are roughly $2W_0T_0$ orthogonal signals of limited duration T_0 and roughly limited bandwidth W_0. The first system needs 2 orthogonal signals of duration 2 seconds while the second needs 4 signals of duration 4 seconds. Thus, both systems could be implemented in the same bandwidth of $W_0 \approx 1/2$ Hz.

(e) The 8-ary system uses orthogonal signaling to transmit 3 bits. Hence, the symbol energy is 3E per symbol, and the "symbol duration" is 6 seconds. Since 8 PPM pulses must fit into 6 seconds, the pulse duration must be 0.75 second, and the pulse amplitude must be $2\sqrt{E}$. Thus, the signal is

$$\sum_{k=-\infty}^{\infty} 2\sqrt{E}\cdot p_{0.75}(t - 6k - (b_{3k} + 2b_{3k+1} + 4b_{3k+2})\cdot 0.75)$$

Note that the 8-ary signal set needs 8 orthogonal signals in 6 seconds, so that the bandwidth required is ≈ 2/3 Hz which is larger than 1/2 Hz! This illustrates the bandwidth expansion required for M-ary orthogonal signaling. Also, the signals (both the 4-ary set as well as the 8-ary set) need more peak power for transmission than binary signaling at the same bit rate. This is a characteristic of PPM signaling.

2.(a) $R_{\mathbf{N}}(t) = \int_{-\infty}^{\infty} S_{\mathbf{N}}(f)\exp(j2\pi ft)df = 2\int_{990}^{1000} 10\cdot\cos(2\pi ft)df + 2\int_{1000}^{1010} 20\cdot\cos(2\pi ft)df = 10\cdot\frac{\sin(2\pi ft)}{\pi t}\Big|_{990}^{1000} + 20\cdot\frac{\sin(2\pi ft)}{\pi t}\Big|_{1000}^{1010}$

$= 10\cdot\frac{\sin(2\pi(1000)t)-\sin(2\pi(990)t)}{\pi t} + 20\cdot\frac{\sin(2\pi(1010)t)-\sin(2\pi(1000)t)}{\pi t}$.

$= 20\cdot[\cos(2\pi(995)t) + 40\cdot\cos(2\pi(1005)t)]\cdot\frac{\sin(2\pi(5)t)}{\pi t} = 200\cdot[\cos(2\pi(995)t) + 2\cdot\cos(2\pi(1005)t)]\cdot\mathrm{sinc}(10t)$.

Quick check: $R_{\mathbf{N}}(0)$ = area under $S_{\mathbf{N}}(f)$ = 600 according to both our formula and the given picture.

An alternative derivation of the same result uses the inverse transform $W\bullet\mathrm{sinc}(Wt) \leftrightarrow \mathrm{rect}(f/W)$. Note that $S_{\mathbf{N}}(f) = (1/2)\bullet[G(f-995) + G(f+995)] + (1/2)\bullet[2\bullet G(f-1005) + 2\bullet G(f+1005)]$ where $G(f) = 20\bullet\mathrm{rect}(f/10)$ has inverse transform $200\bullet\mathrm{sinc}(10t)$. Hence $R_{\mathbf{N}}(t) = 200\bullet[\cos(2\pi(995)t) + 2\bullet\cos(2\pi(1005)t)]\bullet\mathrm{sinc}(10t)$ by the modulation theorem.

(b) With $f_c = 1$ kHz, $S_{\mathbf{X}}(f) = \begin{cases} S_{\mathbf{N}}(f-1000) + S_{\mathbf{N}}(f+1000), & |f| < 1000, \\ 0, & \text{otherwise.} \end{cases}$ Thus, $S_{\mathbf{X}}(f) = 30\bullet\mathrm{rect}(f/20)$.

(c) $R_{\mathbf{X}}(t) \leftrightarrow S_{\mathbf{X}}(f) = 30\bullet\mathrm{rect}(f/20)$. Since $W\bullet\mathrm{sinc}(Wt) \leftrightarrow \mathrm{rect}(f/W)$, we get that $R_{\mathbf{X}}(t) = 600\bullet\mathrm{sinc}(20t)$.

(d)
$$R_{\mathbf{X}}(t) = 2\int_0^{\infty} S_{\mathbf{N}}(f)\cos(2\pi(f-1000)t)df = 2\int_{990}^{1000} 10\bullet\cos(2\pi(f-1000)t)df + 2\int_{1000}^{1010} 20\bullet\cos(2\pi(f-1000)t)df$$
$$= \left.\frac{10\bullet\sin(2\pi(f-1000)t)}{\pi t}\right|_{990}^{1000} + \left.\frac{20\bullet\sin(2\pi(f-1000)t)}{\pi t}\right|_{1000}^{1010} = 600\bullet\mathrm{sinc}(20t) \text{ just as in part (c).}$$

3.(a) The energy in each sinusoid is $(E/T)\bullet(1/2)\bullet 2T = E$. With antipodal signaling and an optimum receiver, the error probability in each branch is thus $Q\left(\sqrt{\frac{2E}{N_0}}\right)$

(b) Since $a_0 = 0$ and $a_1 = 1$, the received signal is $\sqrt{E/T}\bullet[\cos(2\pi f_c t)+\sin(2\pi f_c t)]$ and the signal output in the I branch is $\int_0^{2T}\sqrt{E/T}\bullet[\cos(2\pi f_c t)+\sin(2\pi f_c t)]\bullet 2\bullet\cos(2\pi f_c t+\theta)dt = \sqrt{E/T}\bullet 2T\bullet[\cos\theta - \sin\theta]$ where we have used the identities $2\bullet\cos(A)\bullet\cos(B) = \cos(A+B) + \cos(A-B)$ and $2\bullet\sin(A)\bullet\cos(B) = \sin(A+B) + \sin(A-B)$ and ignored the double-frequency terms. Note that we would have had $\cos\theta+\sin\theta$ if $a_1 = 0$. Similarly, the Q branch signal output is $-\int_0^{2T}\sqrt{E/T}\bullet[\cos(2\pi f_c t)+\sin(2\pi f_c t)]\bullet 2\bullet\sin(2\pi f_c t+\psi)dt = -\sqrt{E/T}\bullet 2T\bullet[\sin\psi + \cos\psi]$.

The noise variance in either branch is $(N_0/2)\int_0^{2T}4\bullet\cos^2(2\pi f_c t+\theta)dt = (N_0/2)\int_0^{2T}4\bullet\sin^2(2\pi f_c t+\theta)dt = 2N_0T$.

Hence, the error probabilities are $Q\left(\sqrt{\frac{2E}{N_0}}\bullet[\cos\theta-\sin\theta]\right)$ in the I branch and $Q\left(\sqrt{\frac{2E}{N_0}}\bullet[\cos\psi+\sin\psi]\right)$ in the Q branch.

4.(a) The signals are orthogonal because the baseband signals $p_{T/2}(t)$ and $p_{T/2}(t-T/2)$ have non-overlapping support, i.e. $s_0(t)s_1(t) = 0$ for all t.

(b) Given that s_0 was transmitted and $\theta = \pi/4$, $\mathbf{U}_0$, $\mathbf{V}_0$, $\mathbf{U}_1$, and $\mathbf{V}_1$ are independent Gaussian random variables with common variance
$$\sigma^2 = \frac{N_0}{2}\int_0^{T/2}4\bullet\cos^2(2\pi f_c t)dt = \frac{N_0}{2}\int_0^{T/2}4\bullet\sin^2(2\pi f_c t)dt = \frac{N_0}{2}\int_{T/2}^{T}4\bullet\cos^2(2\pi f_c t)dt = \frac{N_0}{2}\int_{T/2}^{T}4\bullet\sin^2(2\pi f_c t)dt = \frac{N_0T}{2}.$$
$$E[\mathbf{U}_0] = \int_0^{T/2}\sqrt{2}A\bullet\cos(2\pi f_c t+\pi/4)\bullet 2\bullet\cos(2\pi f_c t)dt = \sqrt{2}A\bullet(T/2)\bullet\cos(\pi/4),\ E[\mathbf{U}_1] = 0 \text{ (no signal in (T/2,T)!!}$$
$$E[\mathbf{V}_0] = \int_0^{T/2}\sqrt{2}A\bullet\cos(2\pi f_c t+\pi/4)\bullet(-2)\bullet\sin(2\pi f_c t)dt = \sqrt{2}A\bullet(T/2)\bullet\sin(\pi/4),\ E[\mathbf{V}_1] = 0 \text{ (no signal in (T/2,T)!!}$$

(c) This is noncoherent demodulation of orthogonal signals, and the error probability is $(1/2)\bullet\exp(-(A_0)^2/4\sigma^2)$ where $(A_0)^2 = (E[\mathbf{U}_0])^2 + (E[\mathbf{V}_0])^2 = 2A^2(T/2)^2 = A^2T^2/2$. Hence, $P_e = (1/2)\bullet\exp(-A^2T/4N_0) = (1/2)\bullet\exp(-E/2N_0)$ since the energy in each pulse of rms amplitude A is $E = A^2(T/2)$.