University of Illinois	Final Exam Tuesday, May 12, 1998 1:30 pm — 4:30 pm	ECE 361 Spring 1998

Instructions:

There are **four problems** on four separate pages on this examination.

The last two pages of the exam are to be torn off and returned with your exam booklet. DON'T FORGET to write your name on each sheet. If you mess up the sheet(s), don't waste time erasing your work. I have extra copies available for your use.

Two pages of notes allowed. No other notes, books, tables of integrals, or calculators/personal computers permitted.

Show your work in the exam booklet provided. Answers without appropriate justification will receive no credit.

Notation: For $\lambda > 0$, $p_\lambda(t) = \begin{cases} 1 & \text{if } 0 \le t < \lambda, \\ 0 & \text{otherwise.} \end{cases}$

$\Phi(x)$ = cumulative probability distribution function for standard (zero-mean unit-variance) Gaussian random variable

$Q(x) = 1 - \Phi(x)$

$P_{e,i}$ = probability of error given that signal $s_i(t)$ was transmitted

AWGN denotes additive (zero-mean) white Gaussian noise with power spectral density $N_0/2$. This random process is independent of the choice of transmitted signal.

Some more-or-less useless facts:

$2\cdot\cos^2(x) = 1 + \cos(2x)$ $\qquad 2\cdot\sin^2(x) = 1 - \cos(2x)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ $\qquad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\text{rect}(t/T) \leftrightarrow T\cdot\text{sinc}(fT)$ and $W\cdot\text{sinc}(Wt) \leftrightarrow \text{rect}(f/W)$ where $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

$\cos(2\pi f_c t) \leftrightarrow \frac{1}{2}\cdot[\delta(f+f_c) + \delta(f-f_c)]$ $\qquad \sin(2\pi f_c t) \leftrightarrow \frac{j}{2}\cdot[\delta(f+f_c) - \delta(f-f_c)]$

Rayleigh density function: $f(r) = \dfrac{r}{\sigma^2}\cdot\exp\left(-\dfrac{r^2}{2\sigma^2}\right)$ $r > 0$

Rician density function: $f(r) = \dfrac{r}{\sigma^2}\cdot I_0\left(\dfrac{rA}{\sigma^2}\right)\cdot\exp\left(-\dfrac{r^2-A^2}{2\sigma^2}\right)$ $r > 0$,

where $I_0(x) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \exp(x\cdot\cos\theta)\,d\theta$

1. Independent data bits, equally likely to be 0 or 1, are to be transmitted at a rate of 1/T bits per second over an AWGN channel. Four different binary finite-duration signal sets of the form shown below are being considered for use in this baseband communication system. As usual, the k-th bit is transmitted using the signal set $s_0(t-kT)$ and $s_1(t-kT)$.

Set I: $s_0(t) = \alpha \cdot p_T(t)$, $s_1(t) = \sqrt{3}\cdot\alpha\cdot(t/T)\cdot p_T(t)$

Set II: $s_0(t) = \sqrt{2}\cdot\beta\cdot\cos(\pi t/T)\cdot p_T(t)$, $s_1(t) = \sqrt{2}\cdot\beta\cdot\sin(\pi t/T)\cdot p_T(t)$

Set III: $s_0(t) = \sqrt{2}\cdot\gamma\cdot\cos(\pi t/T)\cdot p_T(t)$, $s_1(t) = -\sqrt{2}\cdot\gamma\cdot\cos(\pi t/T)\cdot p_T(t)$

Set IV: $s_0(t) = \delta\cdot p_T(t)$, $s_1(t) = -(\delta/\sqrt{2})\cdot p_T(t)$

where α, β, γ, and δ are constants whose value is to be chosen by you so as to obtain the "best" signal set subject to various criteria. *In all cases, assume that the optimum receiver for the chosen signal set is being used.*

Let $E_0 = \int_0^T [s_0(t)]^2 dt$ and $E_1 = \int_0^T [s_1(t)]^2 dt$ denote the signal energies.

(a) Suppose that the transmitter has limited ***average* power** P. Choose α, β, γ, and δ appropriately so that all four signal sets satisfy the constraint

$$\overline{P} = \frac{P_0 + P_1}{2} = \frac{E_0 + E_1}{2T} \le P.$$

Subject to the specified constraint, which of the above signal sets will give the smallest error probability? and what is this smallest error probability?

(b) The condition $\overline{P} = (P_0+P_1)/2 \le P$ of part (a) is unsatisfactory in that one of the signals may very well exceed the power limitation even though the *average* power $\overline{P}$ is no larger than P. Suppose then, that the signal set chosen must satisfy **both** the conditions

$$P_0 \le P \qquad \text{and} \qquad P_1 \le P.$$

Of course, this also implies the constraint $\overline{P} \le P$ similar to the restriction in part (b).

Under these constraints, which of the above signal sets will give the smallest error probability? and what is this smallest error probability?

(c) **Instead of** the average power limitation considered in parts (a) and (b), suppose that the transmitter has a peak ***instantaneous* power** limitation (equivalently, an **amplitude** limitation) so that *for all t*, the chosen signal set must satisfy the constraints

$$|s_0(t)| \le A \qquad \text{and} \qquad |s_1(t)| \le A.$$

Of course, this also implies the constraints: $P_0 \le A^2$ and $P_1 \le A^2$ similar to the restriction in part (b).

Under these constraints, which of the above signal sets will give the smallest error probability? and what is this smallest error probability?

(d) Which of the optimum receivers for the above four signal sets has a threshold value of 0?

Note: *Except trivially in part (d), you are not required to provide any details of the optimum receivers themselves; only to discuss and compare their error probability performance.*

2. A sequence $\{b_k\}$ of independent data bits, equally likely to be 0 or 1, is differentially encoded into the sequence $\{a_k\}$ where $a_k = b_k \quad a_{k-1}$. The sequence $\{a_k\}$ is then transmitted over an AWGN channel via MSK signaling, with the even-numbered bits being transmitted over the inphase carrier and the odd-numbered bits over the quadrature carrier. More specifically, let

$$a_I(t) = \sum_{k=-\infty}^{\infty} (-1)^{a_{2k}} p_{2T}(t-(2k-1)T) \bullet \cos(\pi t/2T) \text{ and } a_Q(t) = \sum_{k=-\infty}^{\infty} (-1)^{a_{2k+1}} p_{2T}(t-2kT) \bullet \sin(\pi t/2T)$$

denote the inphase and quadrature baseband signals and let the MSK signal be given by

$$\sqrt{2E/T} \bullet [a_I(t) \bullet \cos(2\pi f_c t) - a_Q(t) \bullet \sin(2\pi f_c t)].$$

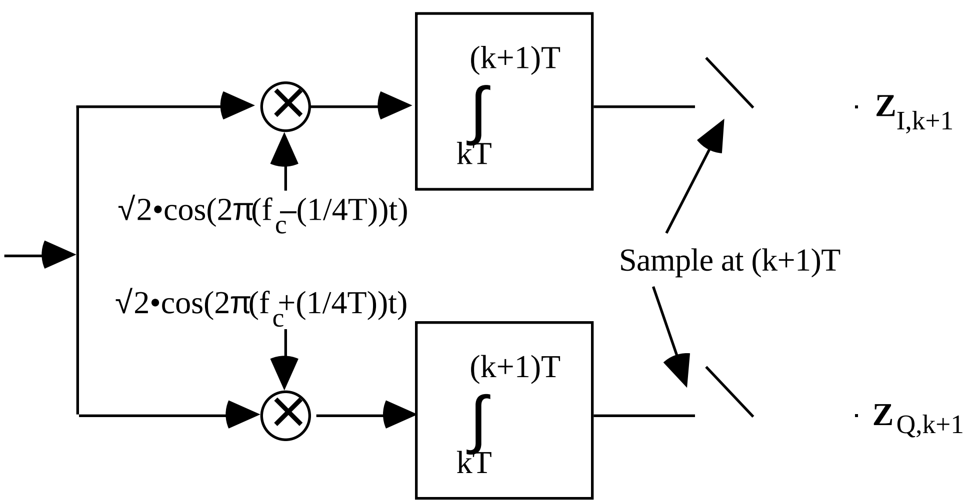
(a) What is the data rate of this MSK signal? What is the signal energy transmitted per bit?

(b) At any time, the frequency of the MSK signal is $f_c \pm 1/(4T)$.

How is the frequency of the MSK signal during the time interval $[kT, (k+1)T)$ related to the **data** bit sequence $\{b_k\}$? Be sure to consider both odd and even values of k.

(c) In the receiver shown below, the integrators are dumped (reset to 0) immediately following the sampling of their outputs. Assume that $f_c T/4$ is a large integer. Note that the outputs at time $(k+1)T$ are $\mathbf{Z}_{I,k+1}$, $\mathbf{Z}_{Q,k+1}$.

Suppose that $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$. Compute the mean and variance of $\mathbf{Z}_{I,1}$, $\mathbf{Z}_{Q,1}$, $\mathbf{Z}_{I,2}$, and $\mathbf{Z}_{Q,2}$. That is, compute the means and variances of the outputs at time t = T and *also at* time t = 2T.



(d) The receiver can make a (suboptimum) decision about the **data** bit b_{k+1} by comparing $|\mathbf{Z}_{I,k+1}|$ with $|\mathbf{Z}_{Q,k+1}|$, or equivalently by comparing $(\mathbf{Z}_{I,k+1})^2$ with $(\mathbf{Z}_{Q,k+1})^2$. It is not necessary to determine the values of the a_k's at all.

If $|\mathbf{Z}_{I,k+1}| > |\mathbf{Z}_{Q,k+1}|$, does the receiver decide that b_{k+1} was 0 or 1?

Note: Since b_{k+1} affects the signal during the interval $[kT,(k+1)T)$, an optimum decision would be based on the four sample values of $\mathbf{Z}_{I,k+1}$, $\mathbf{Z}_{Q,k+1}$, $\mathbf{Z}_{I,k+2}$, $\mathbf{Z}_{Q,k+2}$, not just $\mathbf{Z}_{I,k+1}$ and $\mathbf{Z}_{Q,k+1}$.

(e) What is the error probability of the receiver with the processing as in part (d)?
Careful: this process is **not** noncoherent demodulation.

3. The eight equally likely signals in an 8-PSK communication system operating over an AWGN channel are given by

$$s_k(t) = \sqrt{2E_s/T}\bullet\cos(2\pi f_c t + k\pi/4)\bullet p_T(t), \qquad 0 \le k \le 7.$$

Assume that f_cT is a large integer.

The receiver correlates its input from 0 to T with respectively $\sqrt{2/T}\bullet\cos(2\pi f_c t)$ and $-\sqrt{2/T}\bullet\sin(2\pi f_c t)$ to produce $\underline{\mathbf{Z}} = (\mathbf{Z}_I, \mathbf{Z}_Q)$ where, conditioned on $s_k(t)$ being transmitted, the independent Gaussian random variables $\mathbf{Z}_I$, $\mathbf{Z}_Q$ have means

$$E[\mathbf{Z}_I] = \sqrt{E_s}\bullet\cos(k\pi/4) \text{ and } E[\mathbf{Z}_Q] = \sqrt{E_s}\bullet\sin(k\pi/4)).$$

(a) Conditioned on $s_k(t)$ being transmitted, what is the variance of $\mathbf{Z}_I$ and $\mathbf{Z}_Q$?

(b) The receiver decides that the signal $s_{\hat{k}}(t)$ was transmitted where $\hat{k}$ is the integer, $0 \le \hat{k} \le 7$, that minimizes

$$\left(\mathbf{Z}_I - \sqrt{E_s}\bullet\cos(\hat{k}\pi/4)\right)^2 + \left(\mathbf{Z}_Q - \sqrt{E_s}\bullet\sin(\hat{k}\pi/4)\right)^2$$

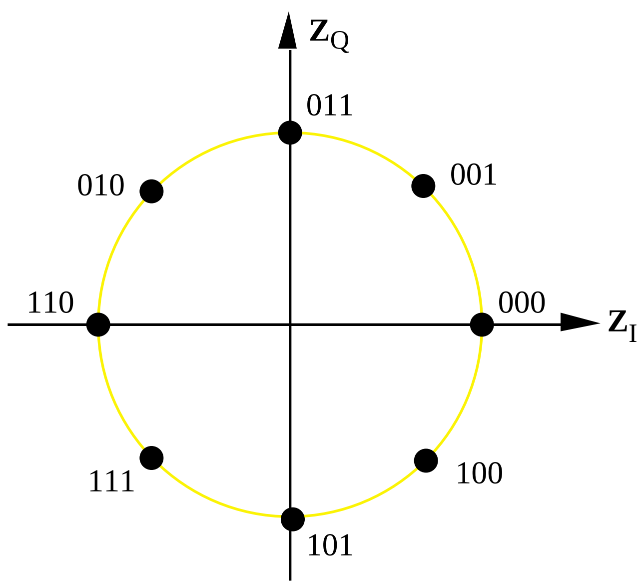
In other words, the receiver sample value $\underline{\mathbf{Z}}$ is demodulated into the nearest signal point.

Show that $P\{\hat{k} \ne k\}$, the symbol error probability, is smaller than $2\bullet Q(\sqrt{2E_s/N_0}\bullet\sin(\pi/8))$.

(c) Assume that the 3 data bits $\underline{\mathbf{b}} = (b_0 b_1 b_2)$ are mapped to the 8-PSK signal set using a Gray code as shown below, and let $\hat{\underline{\mathbf{b}}} = (\hat{b}_0\hat{b}_1\hat{b}_2)$ denote the receiver output. On the copy of this diagram provided to you, indicate

the region A such that $\underline{\mathbf{Z}}$ A implies $\hat{b}_1 = 1$

and the region B such that $\underline{\mathbf{Z}}$ B implies $\hat{b}_2 = 1$



(d) Find $P\{\hat{b}_1 \ne b_1 \mid \underline{\mathbf{b}} = 000\}$ and $P\{\hat{b}_1 \ne b_1 \mid \underline{\mathbf{b}} = 001\}$. Use these results to find $P\{\hat{b}_1 \ne b_1\}$.

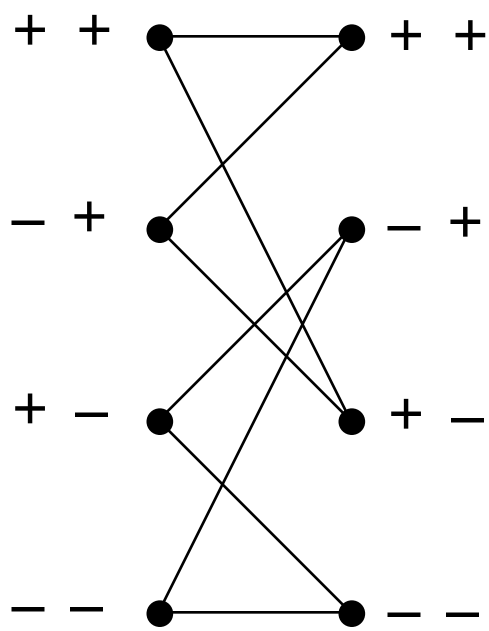
(e) Find $P\{\hat{b}_2 \ne b_2 \mid \underline{\mathbf{b}} = 000\}$ and $P\{\hat{b}_2 \ne b_2 \mid \underline{\mathbf{b}} = 001\}$. Use these results to find $P\{\hat{b}_2 \ne b_2\}$.

4. Maximum-likelihood sequence estimators for intersymbol interference (ISI) channels that use the Viterbi algorithm are usually implemented with special-purpose digital signal processing circuits. A analog-to-digital (A/D) convertor is used to represent the observations using a small number of bits, and integer arithmetic (instead of floating-point arithmetic) is used to simplify the computation of the square of the distance. For example, with 4 bits per observation, the sample values range from –7 to +7, with the "noiseless" signal output peak being ±4 (say) instead of ±1 as used in class. Of course, the accumulated values for the squared distances of the paths are stored using many more than 4 bits in the sample value.

Consider an ISI channel with constraint length 2 in which the signal output (sampled impulse response) is

$$\mathbf{g} = (g_0, g_1, g_2) = (+4, +1, -1).$$

Assume that signaling starts at t = 0 so that the very first channel output has no ISI whatsoever, and the second has ISI only from the previous bit. As usual, the data bits are denoted by + and –, and the basic trellis structure is as shown below.



Four data bits are transmitted and the channel output is (–5, +1, +1, –3, –1, –1).

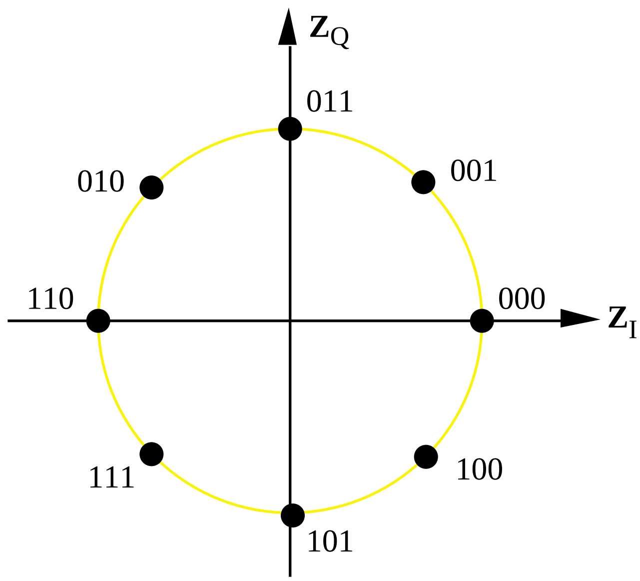
(a) Use the Viterbi algorithm to find the maximum-likelihood estimate of the 4-bit data sequence. **Show your work** on the trellis exhibited on the tear-off page.

(b) Compare the demodulated data sequence to the result that would have been obtained if we had simply ignored ISI and demodulated the data as

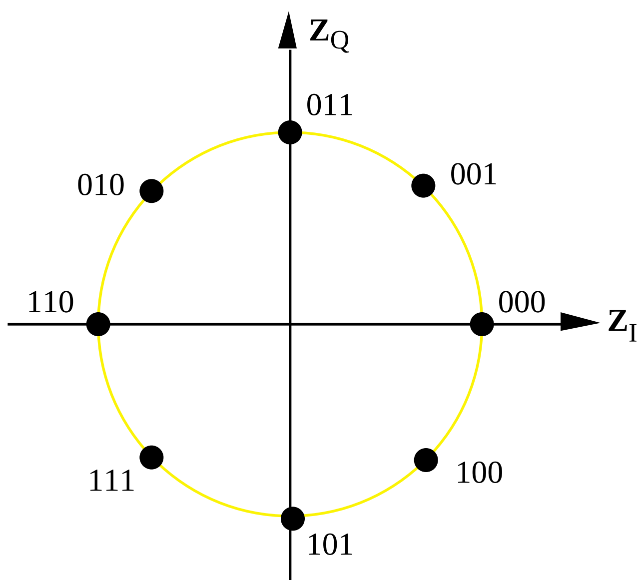
$$\text{sgn}(-5), \text{sgn}(+1), \text{sgn}(+1), \text{sgn}(-3) = - + + -$$

Name:________________________________

3 data bits $\underline{\mathbf{b}} = (b_0 b_1 b_2)$ are mapped to the 8-PSK signal set using a Gray code as shown below, and $\underline{\hat{\mathbf{b}}} = (\hat{b}_0 \hat{b}_1 \hat{b}_2)$ denotes the receiver output. $\underline{\mathbf{Z}} = (\mathbf{Z}_I, \mathbf{Z}_Q)$.



Indicate on this diagram the region A such that $\underline{\mathbf{Z}}$ A implies $\hat{b}_1 = 1$.



Indicate on this diagram the region B such that $\underline{\mathbf{Z}}$ B implies $\hat{b}_2 = 1$.

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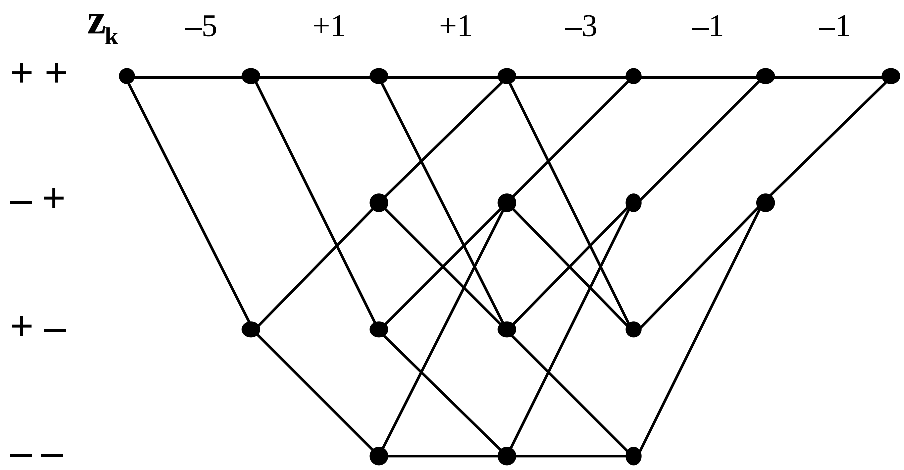
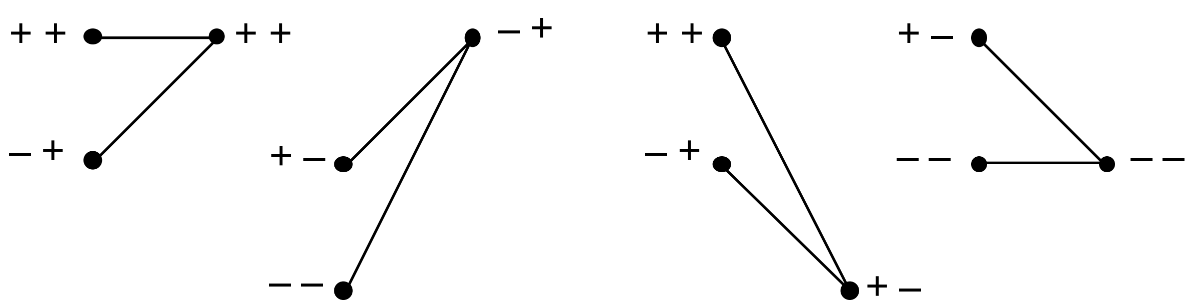


$g_0 = 4$, $g_1 = +1$, $g_2 = -1$



1. The signals are equally likely, and the error probability is $Q\left(\sqrt{\dfrac{E_0 + E_1 - 2\ s_0,s_1}{2N_0}}\right)$

We can find the values of $P_0 = E_0/T$, $P_1 = E_1/T$, $\bar{P} = (P_0 + P_1)/2$, and $E_0 + E_1 - 2\ s_0,s_1$ for each of the signal sets. The results are as follows:

	P_0	P_1	$\bar{P}$	s_0,s_1	$E_0 + E_1 - 2\ s_0,s_1$	
Set I:	α^2	α^2	α^2	$(\sqrt{3}/2)\cdot\alpha^2 T$	$(2-\sqrt{3})\alpha^2 T$	$= (2-\sqrt{3})\cdot\bar{P}T$
Set II:	β^2	β^2	β^2	0	$2\beta^2 T$	$= 2\cdot\bar{P}T$
Set III:	γ^2	γ^2	γ^2	$-\gamma^2 T$	$4\gamma^2 T$	$= 4\cdot\bar{P}T$
Set IV:	δ^2	$\delta^2/2$	$3\delta^2/4$	$-\delta^2 T/\sqrt{2}$	$\delta^2 T(3 + 2\sqrt{2})/2$	$= 2\cdot\bar{P}T\cdot(3+2\sqrt{2})/3$

(a) Clearly, $\alpha^2 = \beta^2 = \gamma^2 = 3\delta^2/4 = \bar{P}$. Since $2-\sqrt{3} < 2 < 2\cdot(3+2\sqrt{2})/3 < 4$, Set III has the smallest error probability which is $Q\left(\sqrt{\dfrac{2\bar{P}T}{N_0}}\right)$

(b) Now we have $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = P$. As in part (a), set III has $E_0 + E_1 - 2\ s_0,s_1 = 4PT$, while set IV now has $E_0 + E_1 - 2\ s_0,s_1 = PT(3+2\sqrt{2})/3 < 2PT < 4PT$. Hence, Set III has the smallest error probability which is $Q\left(\sqrt{\dfrac{2PT}{N_0}}\right)$

(c) We now must equate the maximum amplitudes $\sqrt{3}\alpha$, $\sqrt{2}\beta$, $\sqrt{2}\gamma$, and δ respectively to A. The values of $E_0 + E_1 - 2\ s_0,s_1$ are $(2-\sqrt{3})\cdot A^2T/3$, A^2T, $2\cdot A^2T$, and $[(3+2\sqrt{2})/2]\cdot A^2T$ respectively. Hence, set IV has the least error probability which is $Q\left(\sqrt{\dfrac{A^2T(3+2\sqrt{2})}{4N_0}}\right)$

(d) The threshold is zero if $E_0 = E_1$. Sets I-III satisfy the condition.

2.(a) The bit a_k affects the MSK signal during the interval $[(k-1)T,(k+1)T)$. Note that this overlaps the interval $[kT,(k+2)T)$ during which a_{k+1} affects the MSK signal. Similar observations hold for the data bits b_k. Thus, the data rate is 1/T bits/second.

Over any interval $[kT,(k+1)T)$, the baseband signals $a_I(t)$ and $a_Q(t)$ are $\pm\cos(\pi t/2T)$ and $\pm\sin(\pi t/2T)$, so that the MSK signal is $\pm\sqrt{2E/T}\cdot\cos(2\pi(f_c \pm 1/(4T))t)$. The transmitted energy during $[kT,(k+1)T)$ is thus $2(E/T)\cdot(T/2) = E$ = energy per bit.

(b) Bits a_{2n-1} and a_{2n} affect the MSK signal during $[(2n-1)T,2nT)$. During this interval, the MSK signal is $\sqrt{2E/T}\cdot[a_I(t)\cdot\cos(2\pi f_c t) - a_Q(t)\cdot\sin(2\pi f_c t)]$

$= \sqrt{2E/T}\cdot[(-1)^{a_{2n}}\cdot\cos(\pi t/2T)\cdot\cos(2\pi f_c t) - (-1)^{a_{2n-1}}\cdot\sin(\pi t/2T)\cdot\sin(2\pi f_c t)]$

$= \sqrt{2E/T}\cdot(-1)^{a_{2n}}\cdot[\cos(\pi t/2T)\cdot\cos(2\pi f_c t) - (-1)^{a_{2n-1}}(-1)^{a_{2n}}\cdot\sin(\pi t/2T)\cdot\sin(2\pi f_c t)]$

$= \sqrt{2E/T}\cdot(-1)^{a_{2n}}\cdot[\cos(\pi t/2T)\cdot\cos(2\pi f_c t) - (-1)^{a_{2n-1}\ \ a_{2n}}\cdot\sin(\pi t/2T)\cdot\sin(2\pi f_c t)]$

$= \sqrt{2E/T}\cdot(-1)^{a_{2n}}\cdot[\cos(\pi t/2T)\cdot\cos(2\pi f_c t) - (-1)^{b_{2n}}\cdot\sin(\pi t/2T)\cdot\sin(2\pi f_c t)]$

$= \sqrt{2E/T}\cdot(-1)^{a_{2n}}\cdot\cos(2\pi f_c t + \pi t/2T)$ or $\sqrt{2E/T}\cdot(-1)^{a_{2n}}\cdot\cos(2\pi f_c t - \pi t/2T)$ according as b_{2n} is 0 or 1. Thus, the signal frequency is $f_c + 1/(4T)$ if $b_{2n} = 0$ and $f_c - 1/(4T)$ if $b_{2n} = 1$. The *phase* is 0 or π depending on a_{2n}.

Similarly, bits a_{2n} and a_{2n+1} affect the MSK signal during $[2nT,(2n+1)T)$. During this interval, the MSK signal is $\sqrt{2E/T}\cdot[a_I(t)\cdot\cos(2\pi f_c t) - a_Q(t)\cdot\sin(2\pi f_c t)]$

$= \sqrt{2E/T}\bullet[(-1)^{a_{2n}}\bullet\cos(\pi t/2T)\bullet\cos(2\pi f_c t) - (-1)^{a_{2n+1}}\bullet\sin(\pi t/2T)\bullet\sin(2\pi f_c t)]$
$= \sqrt{2E/T}\bullet(-1)^{a_{2n}}\bullet[\cos(\pi t/2T)\bullet\cos(2\pi f_c t) - (-1)^{a_{2n+1}}(-1)^{a_{2n}}\bullet\sin(\pi t/2T)\bullet\sin(2\pi f_c t)]$
$= \sqrt{2E/T}\bullet(-1)^{a_{2n}}\bullet[\cos(\pi t/2T)\bullet\cos(2\pi f_c t) - (-1)^{a_{2n+1} \oplus a_{2n}}\bullet\sin(\pi t/2T)\bullet\sin(2\pi f_c t)]$
$= \sqrt{2E/T}\bullet(-1)^{a_{2n}}\bullet[\cos(\pi t/2T)\bullet\cos(2\pi f_c t) - (-1)^{b_{2n+1}}\bullet\sin(\pi t/2T)\bullet\sin(2\pi f_c t)]$
$= \sqrt{2E/T}\bullet(-1)^{a_{2n}}\bullet\cos(2\pi f_c t + \pi t/2T)$ or $\sqrt{2E/T}\bullet(-1)^{a_{2n}}\bullet\cos(2\pi f_c t - \pi t/2T)$ according as b_{2n+1} is 0 or 1. Thus, the signal frequency is $f_c + 1/(4T)$ if $b_{2n+1} = 0$ and $f_c - 1/(4T)$ if $b_{2n+1} = 1$. The *phase* is 0 or π depending on a_{2n}.

In summary, the signal frequency during $[kT,(k+1)T)$ depends only on the data bit b_{k+1}, (being equal to $f_c + 1/(4T)$ if $b_{k+1} = 0$ and $f_c - 1/(4T)$ if $b_{k+1} = 1$) but the signal phase can be 0 or π.

(c) Since $\int_{kT}^{(k+1)T} 2\bullet\cos(2\pi(f_c+1/(4T))t)\bullet\cos(2\pi(f_c-1/(4T))t)dt = \int_{kT}^{(k+1)T}\cos(\pi t/T)dt = 0$, $\mathbf{Z}_{I,k+1}$, $\mathbf{Z}_{Q,k+1}$ are (conditonally) independent Gaussian random variables whose means depend on the data symbols, and whose variance is given by $\sigma^2 = (N_0/2)\int_{kT}^{(k+1)T} 2\bullet\cos^2(2\pi(f_c\pm 1/(4T))t)dt = N_0T/2$.

Now, given $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$, we know that $b_1 = 1$ and $b_2 = 0$. From the analysis of part (b) we know that the signal was

$\sqrt{2E/T}\bullet(-1)^{a_0}\bullet\cos(2\pi f_c t - \pi t/2T) = \sqrt{2E/T}\bullet\cos(2\pi f_c t - \pi t/2T)$ during $[0,T)$ and
$\sqrt{2E/T}\bullet(-1)^{a_2}\bullet\cos(2\pi f_c t + \pi t/2T) = -\sqrt{2E/T}\bullet\cos(2\pi f_c t + \pi t/2T)$ during $[T,2T)$.

Hence, $E[\mathbf{Z}_{I,1}] = \sqrt{ET}$, $\mathbf{Z}_{Q,1} = 0 = \mathbf{Z}_{I,2}$, and $\mathbf{Z}_{Q,2} = -\sqrt{ET}$.

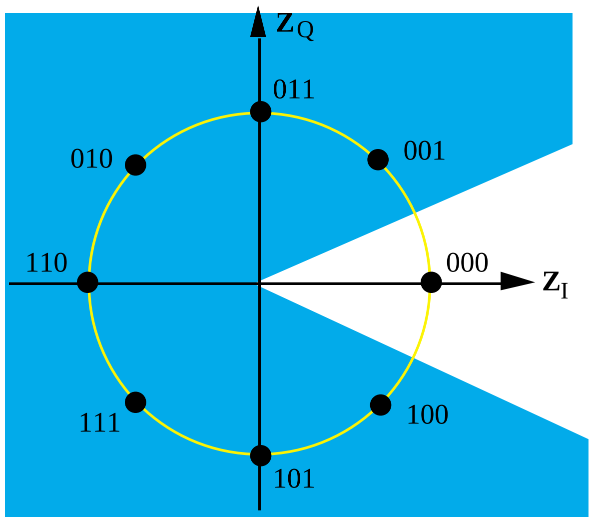
(d) If $b_{k+1} = 0$, $E[\mathbf{Z}_{I,k+1}] = \pm\sqrt{ET}$ while $E[\mathbf{Z}_{Q,k+1}] = 0$. Hence, if $|\mathbf{Z}_{I,k+1}| > |\mathbf{Z}_{Q,k+1}|$, the receiver decides that b_{k+1} was 0.

(e) If $b_{k+1} = 0$, the receiver makes an incorrect decision if $|\mathbf{Z}_{Q,k+1}| > |\mathbf{Z}_{I,k+1}|$, or equivalently, if $(\mathbf{Z}_{Q,k+1})^2 > (\mathbf{Z}_{I,k+1})^2$ where $\mathbf{Z}_{Q,k+1}$ is $N(0,N_0T/2)$ and $\mathbf{Z}_{I,k+1}$ is $N(\pm\sqrt{ET},N_0T/2)$. Thus, the probability of making a correct decision is $P\{(\mathbf{Z}_{I,k+1})^2 > (\mathbf{Z}_{Q,k+1})^2\}$
$= P\{(\mathbf{Z}_{I,k+1}+\mathbf{Z}_{Q,k+1})(\mathbf{Z}_{I,k+1}-\mathbf{Z}_{Q,k+1}) > 0\}$
$= P\{\mathbf{Z}_{I,k+1}+\mathbf{Z}_{Q,k+1} > 0, \mathbf{Z}_{I,k+1}-\mathbf{Z}_{Q,k+1} > 0\} + P\{\mathbf{Z}_{I,k+1}+\mathbf{Z}_{Q,k+1} < 0, \mathbf{Z}_{I,k+1}-\mathbf{Z}_{Q,k+1} < 0\}$
But, $\mathbf{X} = \mathbf{Z}_{I,k+1}+\mathbf{Z}_{Q,k+1}$ and $\mathbf{Y} = \mathbf{Z}_{I,k+1}-\mathbf{Z}_{Q,k+1}$ are independent $N(\sqrt{ET},N_0T)$ or independent $N(-\sqrt{ET},N_0T)$. In either case, $P\{\mathbf{X} > 0, \mathbf{Y} > 0\} + P\{\mathbf{X} < 0, \mathbf{Y} < 0\}$
$= P\{\mathbf{X} > 0\}P\{\mathbf{Y} > 0\} + P\{\mathbf{X} < 0\}P\{\mathbf{Y} < 0\} = [1 - Q(\sqrt{E/N_0})]^2 + Q^2(\sqrt{E/N_0})$ and hence, the error probability is $2\bullet Q(\sqrt{E/N_0})\bullet[1 - Q(\sqrt{E/N_0})]$.

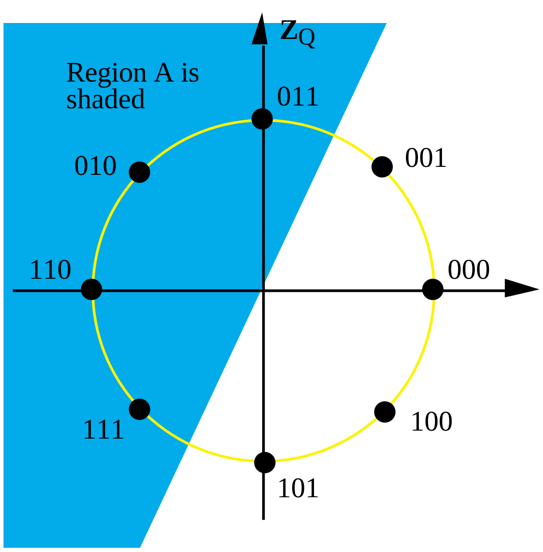
Note: With optimal processing of an MSK signal, the error probability is $Q(\sqrt{2E/N_0})$ which is the error probability for antipodal binary PSK. Instead, we are getting something *related* to $Q(\sqrt{E/N_0})$, the error probability for orthogonal binary FSK. This is reasonable; after all, the two correlators are using orthogonal local references. The exact form of the error probability can be interpreted as due to differential encoding (which is usually a defense against phase ambiguity). After finding $\hat{a}_k$ and $\hat{a}_{k-1}$, we can "determine" $\hat{b}_k$ as $\hat{a}_k \oplus \hat{a}_{k-1}$. Thus, an *error in one o*f $\hat{a}_k$ and $\hat{a}_{k-1}$, and *no error in the other* causes an error in $\hat{b}_k$. Hence, $P\{\hat{b}_k \neq b_k\} = p(1-p) + (1-p)p = 2p(1-p)$. Note that our receiver does not determine $\hat{a}_k$ and $\hat{a}_{k-1}$ separately, but directly finds $\hat{b}_k$. The "phase ambiguity" exhibits itself in the fact that the receiver does not know if $E[\mathbf{Z}_{I,k+1}] = +\sqrt{ET}$ or $-\sqrt{ET}$.

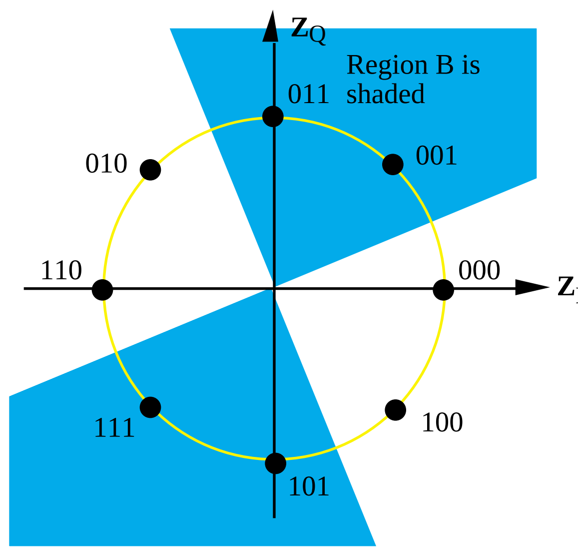
3.(a) $\text{var}(\mathbf{Z}_I) = \text{var}(\mathbf{Z}_Q) = (N_0/2)\int_0^T (2/T)\cdot\cos^2(2\pi f_c t)dt = N_0/2.$

(b) $P\{\hat{k} \neq k\} = P\{\underline{\mathbf{Z}}$ shaded region when 000 is transmitted$\}$
$\leq P\{\mathbf{Z}_Q\cdot\cos(\pi/8) - \mathbf{Z}_I\cdot\sin(\pi/8) > 0\} + P\{\mathbf{Z}_Q\cdot\cos(\pi/8) + \mathbf{Z}_I\cdot\sin(\pi/8) < 0\}$. But, k = 0 and $\mathbf{Z}_Q\cdot\cos(\pi/8) - \mathbf{Z}_I\cdot\sin(\pi/8)$ is Gaussian with mean $-\sqrt{E_s}\cdot\sin(\pi/8)$ and variance $N_0/2$, and similarly $\mathbf{Z}_Q\cdot\cos(\pi/8) + \mathbf{Z}_I\cdot\sin(\pi/8) < 0$ is Gaussian with mean $\sqrt{E_s}\cdot\sin(\pi/8)$ and variance $N_0/2$. Hence, $P\{\hat{k} \neq k\} \leq 2\cdot Q(\sqrt{2E_s/N_0}\cdot\sin(\pi/8))$.



(c) The regions A such that $\underline{\mathbf{Z}}$ A implies $\hat{b}_1 = 1$ and B such that $\underline{\mathbf{Z}}$ B implies $\hat{b}_2 = 1$ are shown below.





(d) From the diagram above and the analysis of part (b), we get
$P\{\hat{b}_1 \neq b_1 \,|\, \underline{\mathbf{b}} = 000\} = Q(\sqrt{2E_s/N_0}\cdot\sin(3\pi/8))$ and
$P\{\hat{b}_1 \neq b_1 \,|\, \underline{\mathbf{b}} = 001\} = Q(\sqrt{2E_s/N_0}\cdot\sin(\pi/8))$. It should be obvious from the symmetry that
$P\{\hat{b}_1 \neq b_1\} = (1/2))\cdot[Q(\sqrt{2E_s/N_0}\cdot\sin(3\pi/8)) + Q(\sqrt{2E_s/N_0}\cdot\sin(\pi/8))]$

(e) Since P(A B) = P(A) + P(B) – 2P(A)P(B) for independent events A and B,
$P\{\hat{b}_2 \neq b_2 \,|\, \underline{\mathbf{b}} = 000\} = P\{\underline{\mathbf{Z}}$ B$\}$

$$= Q\left(\sqrt{\frac{2E_s}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\cdot\sin\left(\frac{3\pi}{8}\right)\right) - 2\cdot Q\left(\sqrt{\frac{2E_s}{N_0}}\cdot\sin\left(\frac{\pi}{8}\right)\right)\cdot Q\left(\sqrt{\frac{2E_s}{N_0}}\cdot\sin\left(\frac{3\pi}{8}\right)\right)$$

$$= P\{\hat{b}_2 \neq b_2 \mid \underline{\mathbf{b}} = 001\} = P\{\hat{b}_2 \neq b_2\}.$$