PROBLEM 7.12 ECE 489 Homework # 2 Solutions

$K = \frac{1}{2} m \dot{x}^2$ Kinetic Energy

$P = m\dot{x} = \frac{dK}{d\dot{x}}$ momentum

For a mechanical system with generalized coords $q_1, \ldots, q_n$

generalized momentum $P_K = \frac{\partial L}{\partial \dot{q}_K} = \frac{\partial K}{\partial \dot{q}_K} - \cancel{\frac{\partial V}{\partial \dot{q}_K}}^{\,0}$

Kinetic Energy $K = \frac{1}{2} \dot{q}^T D(q) \dot{q}$

Lagrangian $L = K - V$

$$\sum_{K=1}^{n} \dot{q}_K P_K = \dot{q}^T P = \dot{q}^T \frac{\partial}{\partial \dot{q}} \left(\frac{1}{2} \dot{q}^T D(q) \dot{q} \right) = \dot{q}^T D(q) \dot{q} = 2K$$

PROBLEM 7.13

(a) $H = \sum_{K=1}^{n} \dot{q}_K P_K - L = \dot{q}^T P - L = \dot{q}^T \left(\frac{\partial L}{\partial \dot{q}} \right) - L$ $\quad P = \frac{\partial L}{\partial \dot{q}}$

$L = K - P = \frac{1}{2} \dot{q}^T D(q) \dot{q} - P(q)$

$\Downarrow$

$$H = \dot{q}^T \left(\frac{\partial}{\partial \dot{q}} \left(\frac{1}{2} \dot{q}^T D(q) \dot{q} - P(q) \right) \right) - \frac{1}{2} \dot{q}^T D(q) \dot{q} + P(q)$$

$$\stackrel{!}{=} \dot{q}^T \left(\frac{1}{2} \left(\dot{q}^T D(q) + D(q) \dot{q} \right) \right) - \frac{1}{2} \dot{q}^T D(q) \dot{q} + P(q)$$

$$\stackrel{!}{=} \dot{q}^T D(q) \dot{q} - \frac{1}{2} \dot{q}^T D(q) \dot{q} + P(q) = \frac{1}{2} \dot{q}^T D(q) \dot{q} + P(q) = K + P$$ TOTAL ENERGY

7.13 (b)

E-L Equations

$$\frac{d}{dt}\frac{\partial}{\partial \dot q_K} L - \frac{\partial L}{\partial q_K} = \tau_K \quad K=1,\dots,n$$

$$H = \sum_{e=1}^{n} \dot q_e p_e - L$$

From E-L

$$\frac{\partial L}{\partial q_K} = \frac{d}{dt}\underbrace{\frac{\partial}{\partial \dot q_K} L}_{p_K} - \tau_K = \frac{d}{dt} p_K - \tau_K$$

$$\left(\frac{\partial L}{\partial q_K} = \underbrace{\frac{\partial}{\partial q_K}\left(\sum_{e=1}^{n} \dot q_e p_e\right)}_{=0} - \frac{\partial H}{\partial q_K}\right) \Rightarrow \frac{\partial L}{\partial q_K} = -\frac{\partial H}{\partial q_K}$$

$$\Downarrow$$

$$-\frac{\partial H}{\partial q_K} = \dot p_K - \tau_K \Rightarrow \boxed{\dot p_K = -\frac{\partial H}{\partial q_K} + \tau_K}$$

$$\frac{\partial H}{\partial p_K} = \frac{\partial}{\partial p_K}\left(\sum_{e=1}^{n} \dot q_e p_e - \frac{\partial L}{\partial p_K}\right) = \dot q_K + \frac{\partial \dot q_K}{\partial p_K} p_K - \left(\frac{\partial K}{\partial p_K} - \cancelto{0}{\frac{\partial V}{\partial p_K}}\right)$$

only K-th component is $\neq 0$

$$= \dot q_K + \frac{\partial \dot q_K}{\partial p_K} p_K - \frac{\partial K}{\partial \dot q_K}\frac{\partial \dot q_K}{\partial p_K} = \dot q_K + \frac{\partial \dot q_K}{\partial p_K} p_K - \underbrace{\frac{\partial L}{\partial \dot q_K}}_{p_K}\frac{\partial \dot q_K}{\partial p_K}$$

$$= \dot q_K$$

$$\boxed{\dot q_K = \frac{\partial H}{\partial p_K}}$$

7.13

(e)

From the textbook pg 259 - 262 (all the quantities are defined in pg. 261)

$\swarrow$ Pg 261

$$K = \frac{1}{2}\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}^T D(q_1, q_2)\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix} \qquad D(q_1, q_2) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{bmatrix}$$

$$P = (m_1 \ell_{c_1} + m_2 \ell_1)\, g \sin(q_1) + m_2 \ell_{c_2}\, g \sin(q_1 + q_2)$$

$$H = K + P \qquad \frac{\partial H}{\partial \dot q} = \underbrace{\frac{\partial L}{\partial \dot q}}_{= p} = \frac{\partial K}{\partial \dot q} = D(q)\,\dot q \Rightarrow \dot q = D^{-1}(q)\, p$$

can be calculated using ~~ROBOTICS TOOLBOX~~ MATHEMATICA

$$\begin{bmatrix} \dot q_1 = \dfrac{\partial H}{\partial p_1} \\ \dot q_2 = \dfrac{\partial H}{\partial p_2}\end{bmatrix} = \begin{bmatrix} D^{-1}(q) \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot p_1 = -\dfrac{\partial H}{\partial q_1} \\ \dot p_2 = -\dfrac{\partial H}{\partial q_2}\end{bmatrix} = -\begin{bmatrix} \dfrac{\partial K}{\partial q_1} + \dfrac{\partial P}{\partial q_1} \\ \dfrac{\partial K}{\partial q_2} + \dfrac{\partial P}{\partial q_2}\end{bmatrix}$$

PROBLEM
7.14

$$\frac{dH}{dt} = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k$$

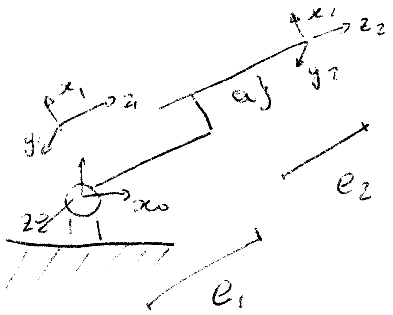
From Hamiltonian's equation

$$\frac{dH}{dt} = \sum_{k=1}^{n} (\tau_k - \dot{p}_k) \dot{q}_k \dot{p}_k$$

$$= \sum_{k=1}^{n} \dot{q}_k \tau_k = \dot{q}^T p$$

The unit of $\frac{dH}{dt}$ are power

PROBLEM ~~[illegible]~~ 4



DH

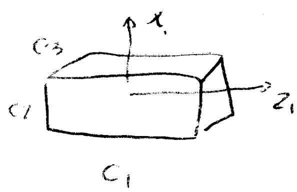
	a	α	d	θ
ℓ_1	a_1	$\pi/2$	0	*
ℓ_2	0	0	*	0

$$COM_1 = \begin{pmatrix} \ell_{c_1} \\ 0 \\ \ell_{c_2} \end{pmatrix}$$

$$COM_2 = \begin{pmatrix} 0 \\ 0 \\ -\ell_{c_3} \end{pmatrix}$$

$$g \; \cancel{[illegible]} = \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix}$$

Inertia tensors



$$I_1 = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

$$I_2 = \begin{bmatrix} I_{xx_2} & & 0 \\ & I_{yy_2} & \\ 0 & & I_{zz_2} \end{bmatrix}$$

$$d_{11} = 2a_1\ell_{c_1}m_1 + a_1^2 m_1 + a_1^2 m_2 + I_{yy} + I_{yy_2} + \ell_{c_1}^2 m_1 + \ell_{c_2}^2 m_1 + m_2\ell_{c_3}^2 - 2m\ell_{c_3}q_2 + m_2 q_2^2$$

$$d_{12} = -a\ell m_2$$

$$d_{21} = d_{12}$$

$$d_{22} = m_2$$

Choose $\theta = [\theta_1 \; \theta_2 \; \theta_3 \; \theta_4 \; \theta_5 \; \theta_6]^T$ (one possible choice)

$$\theta_1 = 2a_1\ell_{c_1}m_1 + a_1^2 m_1 + a_1^2 m_2 + I_{yy} + I_{yy_2} + \ell_{c_1}^2 m_1 + \ell_{c_2}^2 m_1 + m_2\ell_{c_3}^2$$

$$\theta_2 = m_2\ell_{c_3}$$

$$\theta_3 = m_2$$

$$\theta_4 = a_1 m_2$$

$$\theta_5 = (\ell_{c_1} + a_1 m_1)$$

$$\theta_6 = (\ell_{c_2}m_1 - \ell_{c_3}m_2)$$

Then

$D(q)\ddot{q} + C\dot{q} + g =$

$$\begin{bmatrix} \theta_1 \ddot{q}_1 - \theta_2 2 q_2 \dot{q}_2 \dot{q}_1 + \theta_3 q_2^2 \ddot{q}_1 - \theta_4 \ddot{q}_2 - \theta_2 \dot{q}_2 \dot{q}_1 + \theta_3 q_2 \dot{q}_2 \dot{q}_1 - \theta_2 \dot{q}_1 \dot{q}_2 + \theta_3 q_2 \dot{q}_1 \dot{q}_2 + (\theta_4 + \theta_5) g c(q_1) + (\theta_5 + q_2 \theta_3) g s(q_1) \\ -\theta_4 \ddot{q}_1 + \theta_3 \ddot{q}_2 + \theta_2 \dot{q}_1^2 - \theta_3 q_2 \dot{q}_1^2 - \theta_3 g c q_1 \end{bmatrix}$$

$$D(q)\ddot{q} + C\dot{q} + g = Y^T(q, \dot{q}, \ddot{q})\,\theta$$

note the vector of parameters should not contain generalized coordinates

REGRESSOR VECTOR

$$Y(q, \dot{q}, \ddot{q}) = \begin{bmatrix} \ddot{q}_1 & -2q_2\dot{q}_1 - 2\dot{q}_1\dot{q}_2 & q_2^2\ddot{q}_1 + 2q_2\dot{q}_2\dot{q}_1 + q_2 g s(q_1) & -\ddot{q}_2 + g c(q_1) & g c(q_1) + g s(q_1) \\ 0 & \dot{q}_1^2 & -q_2\dot{q}_1^2 - g c(q_1) + \ddot{q}_2 & -\ddot{q}_1 & 0 \end{bmatrix}$$