ECE 489

HW 3

PROBLEM 1.

S skew symm. $\Rightarrow S = -S^T$

$\Downarrow$

$x^T S x = -x^T S^T x \qquad (1)$

scalar

$(x^T S^T x) = (x^T S^T x)^T = x^T S x$

$\Downarrow$

(1) becomes

$(x^T S x) = -x^T S x \Rightarrow \boxed{x^T S x = 0}$

PROBLEM 2.

$$\dot{x}_1 = x_1 - x_1 x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = 2x_1^2 - 2x_2 = f_2(x_1, x_2)$$

<u>equilibrium points:</u>

$$\begin{cases} x_1 - x_1 x_2 = 0 & \longrightarrow \quad x_1(1 - x_2) = 0 \\ 2x_1^2 - 2x_2 = 0 & \longrightarrow \quad x_1^2 - x_2 = 0 \end{cases}$$

$$\Rightarrow \quad x_1 = 0 \quad \text{or} \quad x_2 = 1$$

$$\downarrow \qquad\qquad \downarrow$$

$$x_2 = 0 \qquad\qquad x_1 = \mp 1$$

$\Rightarrow$ $(0,0)$, $(-1,1)$, $(1,1)$ are equilibrium points

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \left(\begin{array}{c|c} 1 - x_2 & -x_1 \\ \hline 4x_1 & -2 \end{array}\right)$$

$$A\big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow \lambda_1 = 1 , \lambda_2 = -2 \quad \rightarrow \text{unstable}$$

$$A\big|_{(-1,1)} = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix} \rightarrow \left.\begin{matrix} \lambda_1 = 1 + 1.73i \\ \lambda_2 = -1 - 1.73i \end{matrix}\right\} \rightarrow \text{stable}$$

$$A\big|_{(1,1)} = \begin{pmatrix} 0 & -1 \\ 4 & -2 \end{pmatrix} \rightarrow \left.\begin{matrix} \lambda_1 = 1 + 1.73i \\ \lambda_2 = -1 - 1.73i \end{matrix}\right\} \rightarrow \text{stable}$$

PROBLEM 3.

(a) $\dot{x}_1 = -x_1 + x_1 x_2^2$

$\dot{x}_2 = -x_2 - x_2 x_1^2$

at equilibrium we have:

$x_1(x_2^2 - 1) = 0$ — (1)

$x_2(x_1^2 + 1) = 0$ — (2)

From (1) $\rightarrow x_1 = 0$ or $x_2 = \mp 1$

From (2) $\rightarrow x_2 = 0$ or $x_1 = \mp j$

(i) assume $x_1 = \mp j$ is an equilibrium $\rightarrow$ by 2 $x_2 = 0$
but then (1) is not satisfied

(ii) assume $x_2 = \mp 1$ is an equilibrium
$\rightarrow$ by (1) $x_1 = 0$, but then (2) is not satisfied

$\Rightarrow$ only case allowed is $(0,0)$ that satisfies both.

b) Let $V = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \rightarrow \frac{dV(x)}{dt} \cdot \dot{x} = x_1 \dot{x}_1 + x_2 \dot{x}_2$

$\dot{V} = x_1(-x_1 + x_1 x_2^2) + x_2(-x_2 - x_1^2 x_2) = -x_1^2 - x_2^2 < 0$

$\Rightarrow$ stable

$\forall (x_1, x_2) \neq (0,0)$

but $x_1 \rightarrow \infty \Rightarrow V(x) \rightarrow \infty \Rightarrow V(x)$ radially unbounded

or

$x_2 \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$

The system is globally asymptotically stable

PROBLEM 4.

$\dot{x} = f(x) + g(x)u \qquad f(0) = 0$

$\exists\ V(x)$ s.t. $V(0)=0$, & $V(x)>0$ & $\frac{\partial V(x)}{\partial x} f(x) \le 0$

$\dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)\, u(x)$

let $u = -\frac{\partial V}{\partial x} g(x)$

$\Rightarrow \dot{V} = \underbrace{\frac{\partial V(x)}{\partial x} f(x)}_{\le 0} - \underbrace{\left(\frac{\partial V(x)}{\partial x} g(x)\right)^2}_{\le 0} \le 0$

required to show that $\dot{V} < 0$

case 1. $\frac{\partial V}{\partial x} f(x) = 0 \Rightarrow \dot{V} = -\left(\frac{\partial V}{\partial x} g(x)\right)^2 < 0$ since $\frac{\partial V}{\partial x} g(x) \neq 0$ if $\frac{\partial V}{\partial x} f(x) = 0$

case 2. $\frac{\partial V}{\partial x} f(x) < 0 \Rightarrow \dot{V} = \overbrace{\frac{\partial V}{\partial x} f(x)}^{<0} - \overbrace{\left(\frac{\partial V}{\partial x} g(x)\right)^2}^{\le 0} < 0$

The only case when $\dot{V} = 0$ is when $\frac{dV}{dx} = 0$ i.e. $x = 0$ since $V(x) > 0$, $V(0) = 0$

Hence the origin is asymptotically stable