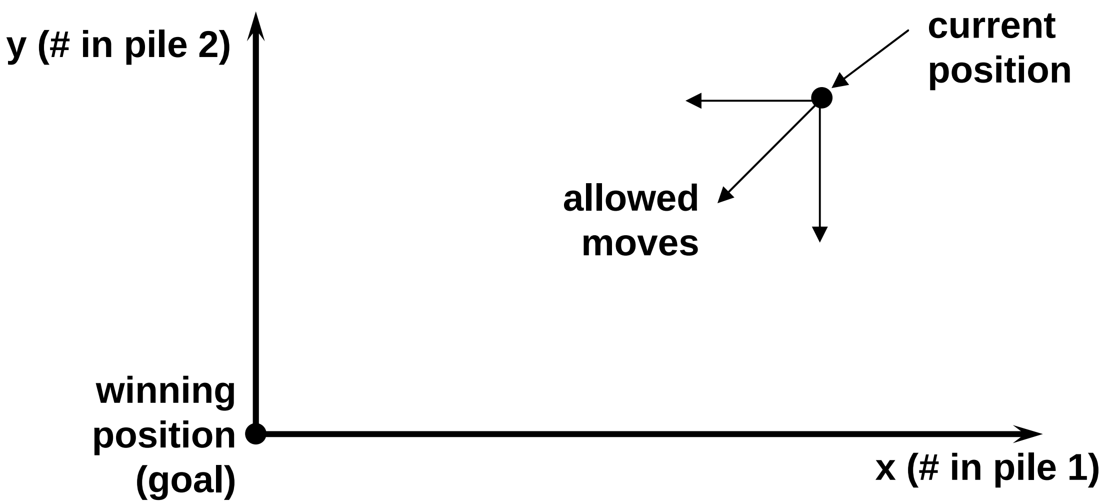
A Sequential Task Specification

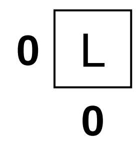
- an example of how hard it is to find a problem with no parallelism
  - taken from…that's right!...ECE190
  - another ECE190 MP3 (Fall 2008), with some simplifications

- a game for two players (Wythoff's game)
  - two piles of sticks
  - goal: take the last stick(s)
  - allowed moves
    - take any # > 0 from either pile
    - take the same # > 0 from both piles

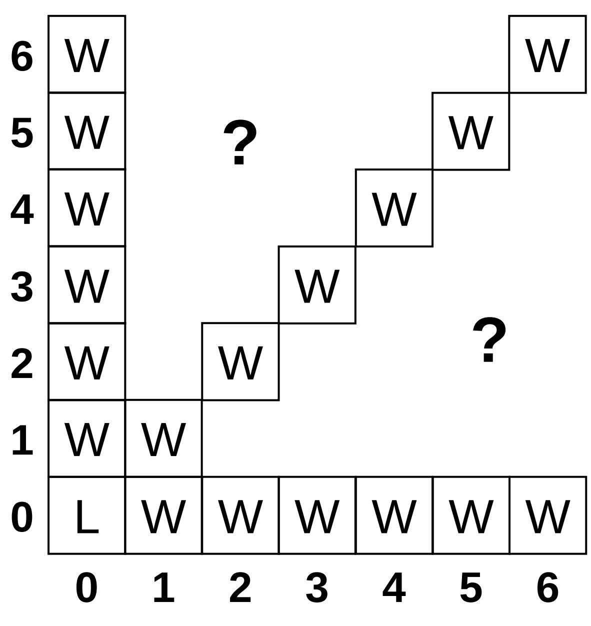


- note
  - no ties, no move cycles, finite # steps to goal (bounded by x + y)
  - thus all positions are forced win/forced loss if played correctly
- question
  - which positions are forced win/loss?
  - (lots of wins; we'll focus on specifying forced loss positions)

- start with the goal state, (0,0)
  - if (x,y) = (0,0) on your turn, you lose
  - thus it's a forced lose



- first set of forced wins
  - any position from which you can reach the goal state in one move
  - for any P > 0
    - (P,0) or
    - (P,P) or
    - (0,P)



- any position
  - for which all possible moves are forced wins
  - is a forced loss
  - thus (1,2) and (2,1) are forced losses

- Which induces more forced wins: for any P > 1,
  - (P+1,1) or
  - (P,2) or
  - (P+1,P) or
  - (P,P+1) or
  - (2,P) or
  - (1,P+1)

6	W	W	W	?		W	W
5	W	W	W		W	W	W
4	W	W	W	W	W	W	
3	W	W	W	W	W		?
2	W	L	W	W	W	W	W
1	W	W	L	W	W	W	W
0	L	W	W	W	W	W	W
	0	**1**	**2**	**3**	**4**	**5**	**6**

- the next forced loss is (3,5) and (5,3), and so forth

- but let’s use induction…

- starting point
  - for simplicity, write forced loss cases as (x,y) with $x < y$
  - observe that any forced loss (x,y) differs by some amount; call it k=y-x
    - base case (0,0) has k=0
    - next couple of cases, (1,2) and (3,5), have k=1, k=2
  - forced loss case for any k is unique
    - assume two cases for some k: (x,x+k) and (z,z+k)
    - assume $x<z$ without loss of generality
    - but (z,z+k) can move to (x,x+k) in one move
      (take z-x from both piles)
    - contradiction: (z,z+k) is not a forced loss, but a forced win!
  - however, forced loss case may not exist for all k

- now use algorithmic induction
  - base case is (0,0); we know it works and thus k=0 case exists
  - assume sequence (x0,y0), (x1,y1), …, (x(k-1), y(k-1)) to some k
  - find method
    - to determine xk and
    - show that (xk, xk + k) is a forced loss

- solution: let xk be
  - the smallest whole number
  - that does not appear in any previous x or y value
  - note: good luck parallelizing the search!

- proof
  - three possible move types from (xk, xk + k)
  - we'll consider one at a time
  - show that all result in forced wins

- first two moves: reduce xk OR reduce both
  - for some p > 0
    - (xk, xk + k) moves to (xk – p, xk + k) OR
    - (xk, xk + k) moves to (xk – p, xk + k – p)
  - by choice of xk, xk – p appears in a previous forced loss
  - thus, for some i, either xi = xk – p or xi + i = xk – p
  - first case
    - forced loss at (xk – p, xk – p + i)
    - note that i < k, so xk – p + i < xk + k – p < xk + k
    - and both (xk – p, xk + k – p) and (xk – p, xk + k) are forced wins
  - second case
    - forced loss at (xk – p – i, xk – p)
    - reverse the indices: clearly xk – p – i < xk + k – p < xk + k
    - again, both (xk – p, xk + k – p) and (xk – p, xk + k) are forced wins

- last move: reduce xk + k
  - for some p > 0, (xk, xk + k) moves to (xk, xk + k - p)
  - first case: |k-p| < k
    - difference k – p is now covered by some previous forced loss case
    - but xk does not appear (by choice of xk) in that case
    - thus result is forced win (forced loss for any k is unique)
  - second case: |k-p| ≥ k (which means p ≥ 2k)
    - reverse indices to put smaller value first: (xk + k – p, xk)
    - xk + k – p < xk, thus xk + k – p appears in a previous forced loss
    - for some i, either xi = xk + k – p or xi + i = xk + k – p
    - first case
      - forced loss at (xk + k – p, xk + k – p + i)
      - again, since i < k, xk + k – p + i < xk
      - and (xk + k – p, xk) is a forced win
    - second case
      - forced loss at (xk + k – p – i, xk + k – p)
      - reverse the indices: clearly xk + k – p – i < xk
      - again, (xk + k – p, xk) is a forced win

- and…
  - there is a closed-form solution!
  - (didn't see one last time I looked, but it's old…maybe as old as 1907?)
  - however, I suggest that you try to find it yourself
  - since without it you can't parallelize…
  - [darn]