University of Illinois
ECE561

Spring 2008
Moulin

Midterm Exam

Tuesday, March 4, 2008

Name SOLUTION SET

Score ______________

Please do not turn this page until requested to do so.

You may use Poor's textbook and your class notes for this exam. Make sure to show all of your work on the longer problems for partial credit. **Box your answers, and write neatly.** Backs of pages may be used for scratch work if necessary.

Problem 1 (15 pts).

(a) (10 pts) Here we model voting as a Bayesian decision problem. Consider an election with two candidates, Haley and Barry. Assume a fixed cost of 1 if voting for Haley. There is some uncertaintly about Barry, which is modeled by an unknown "state" $\theta \geq 0$, and a cost of $2\,e^{-\theta}$ if voting for him. You assume that θ is random with pdf $\alpha e^{-\alpha\theta}$. For what range of values of α will you end up voting for Barry?

Cost for Haley $= 1$

Cost for Barry $= \int_0^\infty 2\,e^{-\theta}\,\alpha\,e^{-\alpha\theta}\,d\theta = \frac{2\alpha}{\alpha+1}\int_0^\infty (\alpha+1)e^{-(\alpha+1)\theta}\,d\theta = \frac{2\alpha}{\alpha+1}$

$\Rightarrow$ Vote for Barry if $1 > \frac{2\alpha}{\alpha+1} \iff \boxed{\alpha < 1}$

(b) (5pts) Your friend agrees with your choice of costs but believes the pdf you have assumed for θ is unrealistic. He will use a minimax strategy for voting. Who will he vote for?

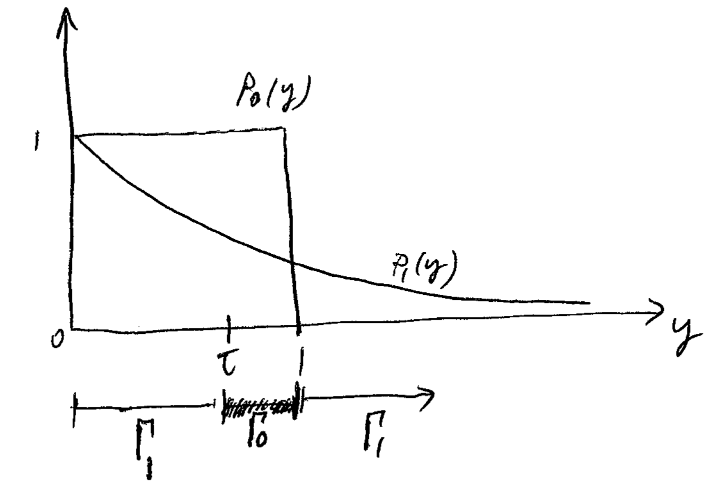
Cost for Haley $= 1$

Max Cost for Barry $= \max_{\theta \geq 0} \left(2\,e^{-\theta}\right) = 2 > 1$

$\Rightarrow$ He'll vote for $\boxed{\text{Haley}}$

Problem 2 (25 pts). Derive the ROC curve for the hypothesis test

$$\begin{cases} H_0: & Y \sim p_0(y) = \text{Uniform } [0,1] \\ H_1: & Y \sim p_1(y) = e^{-y},\ y \geq 0. \end{cases}$$



$$L(y) = \frac{P_1(y)}{P_0(y)} = \begin{cases} e^{-y} & : \ y \leq 1 \\ \infty & : \ \text{else} \end{cases}$$

$$L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \eta \iff \boxed{\Gamma_0 = [\tau, 1] \text{ and } \Gamma_1 = [0, \tau) \cup (1, \infty)}$$

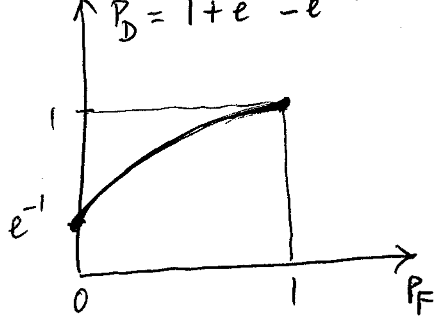
where $0 \leq \tau \leq 1$

$$\alpha = P_F = \int_{\Gamma_1} P_0(y)\,dy = \int_0^\tau dy = \tau$$

$$P_D = \int_{\Gamma_1} P_1(y)\,dy = \int_0^\tau e^{-y}\,dy + \int_1^\infty e^{-y}\,dy$$

$$= 1 - e^{-\tau} + e^{-1}$$

$P_D = 1 + e^{-1} - e^{-P_F}$



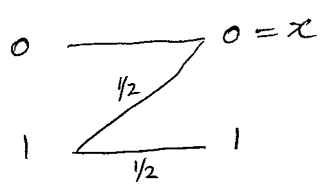
Problem 3 (20 pts). Derive the Neyman-Pearson rules at all significance levels $0 \le \alpha \le 1$ for the binary hypothesis test

$$\begin{cases} H_0: & Y = 0 \\ H_1: & Y \sim p_1(y) = 2^{-(y+1)}, \; y = 0, 1, 2, \cdots. \end{cases}$$

$$L(y) = \frac{P_1(y)}{P_0(y)} = \begin{cases} 2 & \text{if } y = 0 \\ \infty & \text{if } y = 1, 2, 3, \ldots \end{cases}$$

$$\tilde{\delta}(y) = \begin{cases} 1 & \\ \gamma & \text{if } L(y) \overset{>}{\underset{<}{=}} \eta \\ 0 & \end{cases}$$

Observe this problem is very similar to the Z channel problem



where $y = 0 \Rightarrow x = 0$

$y = 1, 2, 3, \ldots \Rightarrow x = 1$

$$\Rightarrow \tilde{\delta}(x) = \begin{cases} 1 & \text{if } x = 1 \\ \alpha & \text{if } x = 0 \end{cases}$$

Equivalently, for our problem, we have

$$\boxed{\tilde{\delta}(y) = \begin{cases} 1 & \text{if } y = 1, 2, 3, \ldots \\ \alpha & \text{if } y = 0 \end{cases}}$$

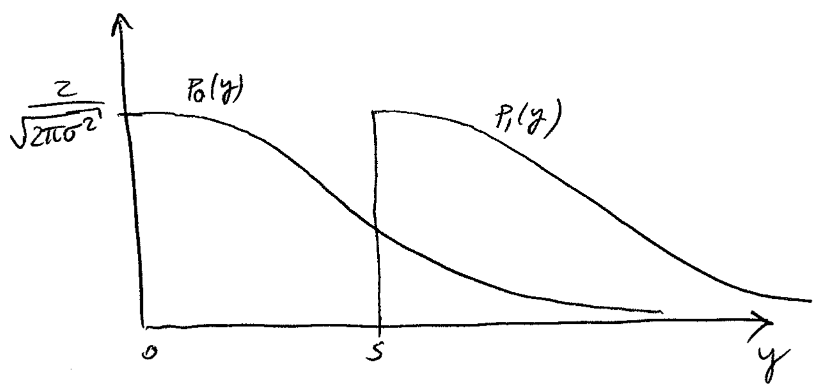
Problem 4 (40 pts). Consider the following hypothesis test:

$$\begin{cases} H_0: & Y = N \\ H_1: & Y = S + N \end{cases}$$

where $S > 0$, and N is a random variable distributed as a "one-sided Gaussian":

$$p_N(n) = \begin{cases} \frac{2}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{n^2}{2\sigma^2}\} & : n \geq 0 \\ 0 & : \text{else.} \end{cases}$$

(a) (15 pts) Give the Bayes rule for deciding between H_0 and H_1 assuming known S, equal priors, and uniform costs.



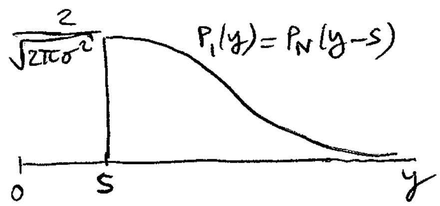
$$L(y) = \frac{P_1(y)}{P_0(y)} = \frac{P_N(y-S)}{P_N(y)}$$

- For $y < S$: $L(y) = 0 \Rightarrow \boxed{\text{Say } H_0}$
- For $y \geq S$: $L(y) = e^{-\frac{(y-S)^2}{2\sigma^2} + \frac{y^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta = 1$

$$\underbrace{-2yS + S^2}_{\substack{\text{always} < 0 \\ \text{because } y \geq S}} \underset{H_1}{\overset{H_0}{\gtrless}} 0 \Rightarrow \boxed{\text{Say } H_1}$$

$$\Rightarrow \boxed{y \underset{H_0}{\overset{H_1}{\gtrless}} S}$$

(b) (15 pts) Now assume S is unknown, and derive the GLRT for this problem.



$$\hat{s}_{ML} = \underset{s \geq 0}{\operatorname{argmax}} \; P_N(y-s)$$
$$= \underset{s \geq y}{\operatorname{argmin}} \; (y-s)^2$$
$$= y$$

$$\Rightarrow P_N(y - \hat{s}_{ML}) = P_N(0) = \frac{2}{\sqrt{2\pi\sigma^2}}$$

$$\Rightarrow L_G(y) = \frac{P_N(y-\hat{s}_{ML})}{P_N(y)} = e^{\frac{y^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\boxed{y \underset{H_0}{\overset{H_1}{\gtrless}} \sigma\sqrt{2\ln\eta}}$$

(c) (10 pts) Is the GLRT a UMP test here?

From Part (a) using arbitrary LRT threshold η,
we see that NP test is of the form

$$\boxed{y \underset{H_0}{\overset{H_1}{\gtrless}} \tau} \quad \text{where } \alpha = 2\,Q\left(\frac{\tau}{\sigma}\right), \quad \underline{\text{for any } s > 0.}$$

$\Rightarrow$ $\boxed{\text{GLRT of Part (b) is UMP}}$