University of Illinois
ECE561

Spring 2008
Moulin

Midterm Exam #2

Tuesday, April 8, 2008

Name SOLUTION SET

Score ____________

Please do not turn this page until requested to do so.

You may use Poor's textbook and your class notes for this exam. Make sure to show all of your work on the longer problems for partial credit. **Box your answers, and write neatly.** Backs of pages may be used for scratch work if necessary.

There are 3 problems, for a total of 110 points.

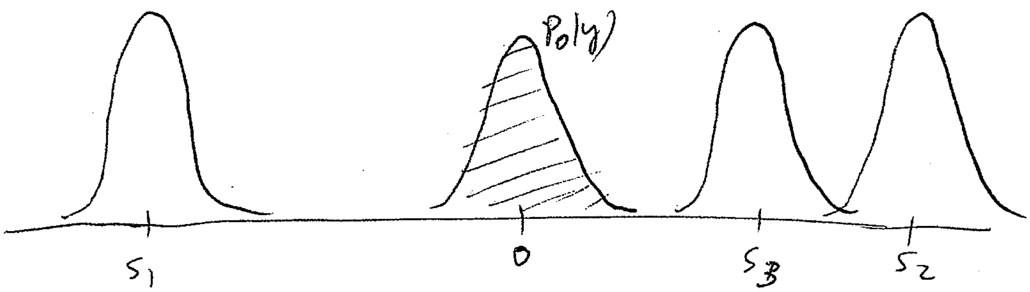
Problem 1 (30 pts). Let S_i, $1 \le i \le n$, be a signal to be optimally designed, N_i, $1 \le i \le n$, an iid $\mathcal{N}(0,1)$ noise sequence, and T a *sample selector* taking values in $\{1, 2, \cdots, n\}$. Consider the following composite hypothesis test:

$$\begin{cases} H_0: & Y = N_T, \\ H_1: & Y = S_T + N_T, \quad T \in \{1, 2, \cdots, n\}. \end{cases}$$

That is, the received Y is a single sample (acquired at a location T unknown to the detector) of the noise sequence or of the signal-plus-noise sequence.

Identify the signals S_i, $1 \le i \le n$, that maximize the power of the detection test at significance level α, for the worst-case T. The design is subject to the energy constraint $\sum_{i=1}^{n} S_i^2 \le n$.

The rival pdf's are $\begin{cases} H_0: & Y \sim \mathcal{N}(0,1) \\ H_1: & Y \sim \mathcal{N}(S_T, 1) \end{cases}$



If $\underbrace{\text{all } S_i > 0}$ OR $\underbrace{\text{all } S_i < 0}$, we have a UMP test:

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \tau \qquad\qquad y \underset{H_1}{\overset{H_0}{\gtrless}} \tau = Q^{-1}(\alpha)$$

The worst-case T is the one for which $|S_T|$ is smallest.

$\Rightarrow$ "Equalizer designs" $\boxed{S_1 = S_2 = \cdots = S_n = 1}$

and $\boxed{S_1 = S_2 = \cdots = S_n = -1}$.

Satisfy energy constraint with equality and maximize power of the test for worst-case T:

$$\min_T P_D = 1 - \max_T Q(|S_T - \tau|) = \boxed{1 - Q(1-\tau)}$$

Problem 2 (40 pts). Let N_i, $1 \le i \le n$, be iid Bernoulli random variables with $Pr[N=1] = \theta \le \frac{1}{2}$. Consider the following binary hypothesis test:

$$\begin{cases} H_0: & Y_i = N_i, \\ H_1: & Y_i = S_i \oplus N_i, \quad 1 \le i \le n, \end{cases}$$

where $\underline{S} = \{S_i, 1 \le i \le n\}$, is a fixed signal, and $\oplus$ denotes modulo-2 addition.

(a) (20 pts) Assuming equal priors and uniform costs, derive the Bayes rule for this test.

Let $\tilde{n} = \sum_{i=1}^{n} 1_{\{S_i=1\}}$ = Hamming weight of $\underline{S}$. $\le n$

$I = \{i : s_i = y_i = 1\}$

$\tilde{n}_1 = |I| \le \tilde{n}$

With this notation we have

LRT: $$\frac{P_1(\underline{y})}{P_0(\underline{y})} = \frac{(1-\theta)^{\tilde{n}_1}\, \theta^{\tilde{n}-\tilde{n}_1}}{\theta^{\tilde{n}_1}\,(1-\theta)^{\tilde{n}-\tilde{n}_1}} = \left(\frac{1-\theta}{\theta}\right)^{2\tilde{n}_1 - \tilde{n}} \underset{H_0}{\overset{H_1}{\gtrless}} \tau = 1$$

$$\Rightarrow \tilde{n}_1 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\left(\tilde{n} + \frac{\ln \tau}{\ln \frac{1-\theta}{\theta}}\right) \Rightarrow \boxed{\tilde{n}_1 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\tilde{n}}$$

(b) (10 pts) Determine the signal $\underline{S}$ that minimizes the error probability of the test above.

$P_e = P_0\left[\tilde{N}_1 > \frac{1}{2}\tilde{n}\right]$ is minimized when $\tilde{n} = n$,

i.e., $\boxed{S_1 = S_2 = \dots = S_n = 1}$ (all-one sequence)

(c) (10 pts) Give a large-deviations upper bound on the error probability above.

$P_e \le e^{-n\mu^*(1/2)}$ where $\mu^*(a) = a \log \frac{a}{\theta} + (1-a)\log\frac{1-a}{1-\theta}$, $0<a<1$

(HW 4 Prob 4(a))

$\Rightarrow \mu^*(\frac{1}{2}) = -\frac{1}{2}\log\left[4\theta(1-\theta)\right]$

$\Rightarrow \boxed{P_e \le \left[4\theta(1-\theta)\right]^{n/2}}$

Problem 3 (40 pts). Let $X_i, 1 \le i \le n$, be iid random variables with unit exponential distribution.

(a) (20 pts) Use a large-deviations method to derive an upper bound on $Pr[\underbrace{\sum_{i=1}^{n} X_i}_{=\sigma_n} > n^\alpha]$, where $\alpha > 1$.

The c.g.f. for $\sigma_n = \sum_{i=1}^{n} X_i$ is $\mu_{\sigma_n}(s) = n\mu_X(s) = n \log \frac{1}{1-s}$ for $s<1$

The large-deviations bound is (observe $n^\alpha > E[\sigma_n] = n$ for $\alpha>1$)

$$Pr[\sigma_n > n^\alpha] \le e^{-[s n^\alpha - \mu_{\sigma_n}(s)]}$$

$$= e^{-n[s \underbrace{n^{\alpha-1}}_{=a} - \log \frac{1}{1-s}]}$$

$\Rightarrow$ from HW 4 Pbm 4(b) we obtain

$$Pr[\sigma_n > n^\alpha] \le e^{-n\mu^*(n^{\alpha-1})}$$

where $\mu^*(a) = a - \log a - 1$ for $a>0$

$$= e^{-n[n^{\alpha-1} - (\alpha-1)\log n - 1]}$$

$$= \boxed{e^{-[n^\alpha - (\alpha-1)\, n \log n - n]}}$$

Since $\alpha > 1$, the dominating term in the exponent is n^α, hence $Pr[\sigma_n > n^\alpha]$ vanishes superexponentially with n.

(b) (10 pts) Use the Central Limit Theorem to approximate $Pr[\underbrace{\sum_{i=1}^{n} X_i}_{=\sigma_n} > n+n^{\alpha}]$, where $\frac{1}{2} \le \alpha < 1$.

$$E[X] = Var[X] = 1 \quad \Rightarrow \begin{cases} E[\sigma_n] = n \\ Var[\sigma_n] = n \end{cases}$$

$$\Rightarrow \hat{P}_{CLT,n} = Q\left(\frac{n^{\alpha}}{\sqrt{n}}\right) = \boxed{Q(n^{\alpha-1/2})}$$

As $n \to \infty$, we have $\hat{P}_{CLT,n} \sim \frac{1}{\sqrt{2\pi n^{\alpha-1/2}}} e^{-\frac{1}{2}n^{2\alpha-1}}$

(c) (10 pts) Use the large-deviations method of Part (a) to derive an upper bound on the tail probability of Part (b). You may use asymptotic approximations, e.g., $\ln(1+\epsilon) \sim \epsilon - \frac{1}{2}\epsilon^2$ as $\epsilon \to 0$. Compare your result with that of Part (b).

Following the steps of Part (a), we obtain (observe $n+n^{\alpha} > E[\sigma_n] = n$)

$$Pr\Big[\underbrace{\sum_{i=1}^{n} X_i}_{=\sigma_n} > n+n^{\alpha}\Big] \le e^{-[s(n+n^{\alpha}) - \mu_{\sigma_n}(s)]}$$

$$= e^{-n\left[s\underbrace{(1+n^{\alpha-1})}_{a\downarrow 1} - \log\frac{1}{1-s}\right]}$$

$$\Rightarrow Pr[\sigma_n > n+n^{\alpha}] \le e^{-n\mu^*(1+n^{\alpha-1})}$$

where $\mu^*(a) = a - \ln a - 1$

$\sim \frac{1}{2}(a-1)^2$ as $a \to 1$

$\mu^*(1+n^{\alpha-1}) \sim \frac{1}{2}n^{2\alpha-2}$

$$\lesssim \boxed{e^{-\frac{1}{2}n^{2\alpha-1}}}$$

The CLT approximation of Part (b) correctly captures the exponent.

<u>NOTE</u>. The regime $\frac{1}{2} \le \alpha < 1$ is known as the "moderate-deviations" regime (Dembo).