University of Illinois **Spring 2005**
ECE561 **Moulin**

Final Exam

Saturday, May 7, 2005

Name ________________________________

Score __________

Please do not turn this page until requested to do so.

This is a closed-book exam; you may however use three cheat sheets. Make sure to show all of your work on the longer problems for partial credit. Box your answers, and write neatly. Backs of pages may be used for scratch work if necessary. No calculators are allowed.

Problem 1 (20 pts). Consider a point $\mathbf{s}$ on the unit circle in $\mathbb{R}^2$. You are given n iid points

$$\mathbf{Y}_i = \mathbf{s} + \mathbf{N}_i, \quad 1 \le i \le n,$$

where each $\mathbf{N}_i \in \mathbb{R}^2$ is Gaussian with mean zero and covariance matrix $C = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where $|\rho| < 1$.

(a) (15 pts) Determine the ML estimator of $\mathbf{s}$.

(b) (5 pts) Is this estimator a sufficient statistic for estimating $\mathbf{s}$? A minimal sufficient statistic? Explain your answer.

Problem 2 (20 pts). Consider the hypothesis test

$$\begin{cases} H_0 & : \mathbf{Y}_i = \mathbf{N}_i, & 1 \leq i \leq n \\ H_1 & : \mathbf{Y}_i = \mathbf{s} + \mathbf{N}_i, & 1 \leq i \leq n \end{cases}$$

where $\mathbf{s}$ is on the unit circle in $\mathbb{R}^2$ (you don't know where), and the vectors $\mathbf{N}_i$ are iid normal.

(a) (10 pts) Is there a Uniformly Most Powerful test for this problem? Explain why.

(b) (10 pts) Derive the Generalized Likelihood Ratio Test for this problem, and sketch the boundary of the decision region.

Problem 3 (20 pts) Consider the binary hypothesis testing problem

$$\begin{cases} H_0 & : Y_k = N_k, \qquad\quad 1 \le k \le 10 \\ H_1 & : Y_k = S_k + N_k, \quad 1 \le k \le 10 \end{cases}$$

where N_k are independent Gaussian noise samples with mean zero and variances k.

(a) (10 pts) Design the signal $S_k, 1 \le k \le 10$ so as to minimize probability of error, subject to the energy constraint $\sum_{k=1}^{10} S_k^2 \le E$.

(b) (10 pts) It is known that one sample from the sequence $Y_k, 1 \le k \le 10$ might be erased, so the detector might have access to only 9 samples. The location of any missing sample is known to the detector, but unknown to the signal designer. Repeat the optimal signal design problem of Part (a) to minimize the maximum (over all possible erasures) probability of error.
Hint. There are two equally good solutions.

Problem 4 (20 pts) The random variable W is equal to 0 with probability $1-\epsilon$, and to a $\mathcal{N}(0,1)$ random variable with probability ϵ. Given a length-n iid sequence $\mathbf{W}$ generated from this distribution and a known signal $\mathbf{s}$, consider the binary hypothesis test

$$\begin{cases} H_0 & : \mathbf{Y} = -\mathbf{s} + \mathbf{W} \\ H_1 & : \mathbf{Y} = \mathbf{s} + \mathbf{W} \end{cases}$$

Derive an upper bound on the probability of error for that test; the bound should be tight in the exponent as $n \to \infty$.
Hint #1: Recall the definition of the Dirac impulse: $\int f(x)\delta(x)\,dx = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} f(x)\,dx$.
Hint #2: The Chernoff distance between $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(\mu,\sigma^2)$ is equal to $\frac{t(1-t)\mu^2}{2\sigma^2}$ for all Chernoff exponents $0 < t < 1$.

Problem 5 (20 pts) Let $Y_i = \theta + W_i$, $1 \le i \le n$, where W_i are iid Laplacian random variables with variance σ^2. The Laplacian pdf is given by $p_W(w) = (\sqrt{2}\sigma)^{-1} \exp\{-\sqrt{2}|w|/\sigma\}$.

(a) (15 pts) Prove that for any constant $c > 0$ we have

$$\lim_{n\to\infty} Pr\left[|\text{med}\{Y_1, \cdots, Y_n\} - \theta| \ge n^{-1/2}c\right] \le \exp\left\{-\frac{c^2}{\sigma^2}\right\}$$

where med$\{\cdots\}$ denote the median value of a set of numbers.

(b) (5 pts). Is it also true that

$$\lim_{n\to\infty} \exp\left\{\frac{nc^2}{\sigma^2}\right\} Pr\left[|\text{med}\{Y_1, \cdots, Y_n\} - \theta| \ge c\right] \le 1?$$

Explain why.