University of Illinois
ECE561

Spring 2005
Moulin

Midterm Exam
Wednesday, April 6, 2005

Name SOLUTION SET

Score ________

Please do not turn this page until requested to do so.

This is a closed-book exam; you may however use two cheat sheets. Make sure to show all of your work on the longer problems for partial credit. Box your answers, and write neatly. Backs of pages may be used for scratch work if necessary. No calculators are allowed.

Problem 1 (20 pts). A variety of pdf's can be expressed in the following form:

$$p_\theta(y) = C(\theta)h(y)e^{\theta y}.$$

Derive the cumulant-generating function for such pdf's.

C.g.f.

$$\begin{aligned}
\mu(s) &\triangleq \ln \int p_\theta(y)\, e^{sy}\, dy \\
&= \ln \int C(\theta)\, h(y)\, e^{\theta y} e^{sy}\, dy \\
&= \ln \int C(\theta)\, h(y)\, e^{(\theta+s)y}\, dy \\
&= \ln \left[\frac{C(\theta)}{C(\theta+s)} \underbrace{\int C(\theta+s)\, h(y)\, e^{(\theta+s)y}\, dy}_{=1} \right] \\
&= \boxed{\ln \frac{C(\theta)}{C(\theta+s)}}
\end{aligned}$$

Problem 2 (**40** pts). The Poisson distribution with parameter $\lambda > 0$ is given by $p_N(n) = \frac{1}{n!}\lambda^n e^{-\lambda}$ for $n = 0, 1, 2, \cdots$. Its cumulant-generating function is $\mu(s) = \lambda(e^s - 1)$.

(a) (**20** pts) Derive the large-deviation function for p_N. Show it may be written as $\mu^*(a) = \lambda\phi(a/\lambda)$; identify and sketch the function ϕ. Indicate the values of $\phi(0)$, $\phi(1/e)$, $\phi(1)$, $\phi(e)$, and $\phi(\infty)$.

Large-deviation function:

$$\mu^*(a) = \max_{s \in \mathbb{R}} \left[sa - \mu(s) \right]$$

$$= \max_{s \in \mathbb{R}} \left[sa - \lambda(e^s - 1) \right]$$

Maximizer s^* satisfies $0 = \frac{d[\cdots]}{ds} = a - \lambda e^{s^*} \Rightarrow s^* = \ln \frac{a}{\lambda}$

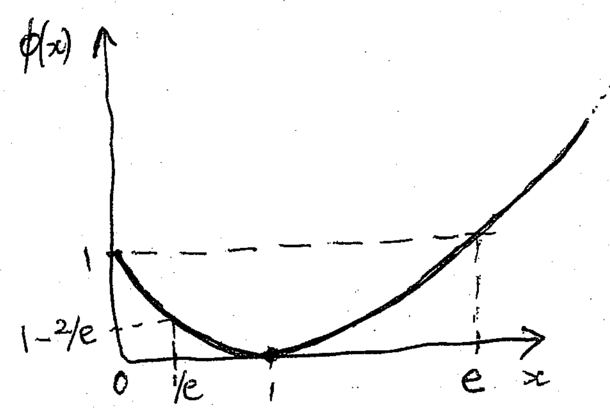
$$\Rightarrow \mu^*(a) = s^* a - \lambda(e^{s^*} - 1)$$

$$= a \ln \frac{a}{\lambda} - a + \lambda$$

$$= \lambda \left(\frac{a}{\lambda} \ln \frac{a}{\lambda} - \frac{a}{\lambda} + 1 \right)$$

$$= \lambda \phi\left(\frac{a}{\lambda}\right)$$

where $\boxed{\phi(x) = x \ln x - x + 1 \quad \text{for } x \geq 0}$



(c) (20 pts) Give a large-deviation upper bound on $Pr[N \geq e\lambda]$ (where $e \approx 2.718$). Comment on the tightness of this bound for $\lambda = 1$ and for $\lambda = 10^8$.

Hint. If $N_1, \cdots, N_m$ are independent and Poisson-distributed random variables with respective parameters $\lambda_1, \cdots, \lambda_m$, then $\sum_{i=1}^m N_i$ is also Poisson distributed, with parameter $\sum_{i=1}^m \lambda_i$.

Poisson $\Rightarrow E[N] = \mu'(0) = \lambda$ $\qquad\qquad \phi(x) = x \ln x - x + 1$

$$\Pr[N \geq e\lambda] \leq e^{-\mu^*(e\lambda)} = e^{-\lambda \phi(e)} = \boxed{e^{-\lambda}}$$

- For $\lambda = 1$, the upper bound is $e^{-1} \approx \frac{1}{2.718}$

  The actual value of $\Pr[N \geq e)$ is $1 - [\Pr[N=0] + \Pr[N=1] + \Pr[N=2]]$

  $$= 1 - e^{-1}\left(1 + 1 + \tfrac{1}{2}\right)$$

  $\left(P_N(n) = \frac{1}{n!} e^{-1}\right)$

  $$\approx 1 - \frac{2.5}{2.718}$$

  $$= \frac{0.218}{2.718}$$

  $\Rightarrow$ above 5 times smaller than upper bound.

- For $\lambda = 10^8$, we may think of N as the sum of many iid Poisson random variables. By application of Cramer's theorem, the large-deviation bound is tight in the exponent;

  $$-\frac{1}{\lambda} \ln \Pr[N \geq e\lambda] \approx -1.$$

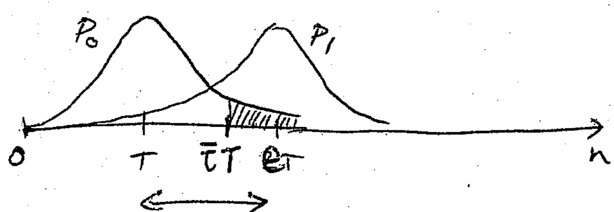
  However, the bound on $\Pr[\ldots]$ itself is loose.

Problem 3 (40 pts). The following problem is encountered in optical communications. A device counts N photons during T seconds and decides whether the observations are due to background noise or to signal plus noise. The hypothesis test is of the form

$$H_0 : N \sim \text{Poisson}(T) \quad \text{versus} \quad H_1 : N \sim \text{Poisson}(eT).$$

We'd like to design the detector to minimize the probability of miss P_M subject to an upper bound $P_F \leq e^{-\alpha T}$ on the probability of alarm. Here $\alpha > 0$ is a tradeoff parameter. When T is large, we can use large-deviations bounds to approximate the optimal test.

Derive a large-deviations bound on P_M, and determine the range of α in which your bound is useful. Sketch the corresponding curve $-\frac{1}{T}\ln P_M$ *vs* $-\frac{1}{T}\ln P_F$ and identify its endpoints.



↳ useful range for threshold of NP test: $n \underset{H_0}{\overset{H_1}{\gtrless}} \bar\tau \cdot T$

where $1 < \bar\tau < 2$

$$\text{From Pbm 2: } \overbrace{\underset{\downarrow \atop \text{Poisson}(T)}{P_0}}^{(P_F =)}[N \geq \bar\tau T] \leq e^{-T\phi(\bar\tau)} \Rightarrow \text{need } \phi(\bar\tau) = \alpha = -\tfrac{1}{T}\ln P_F$$

$$\Rightarrow \boxed{\bar\tau = \phi^{-1}(\alpha)}$$

$$\text{Similarly, } P_M = \underset{\downarrow \atop \text{Poisson}(eT)}{P_1}[N \leq \bar\tau T] \leq e^{-Te\,\phi\left(\frac{\bar\tau}{e}\right)} \Rightarrow \boxed{-\tfrac{1}{T}\ln P_M = e\,\phi\left(\tfrac{\bar\tau}{e}\right)}$$

$$\text{Extreme cases: } \bar\tau = 1 \Rightarrow \begin{cases} -\frac{1}{T}\ln P_F = \phi(1) = 0 \\ -\frac{1}{T}\ln P_M = e\,\phi(1/e) = e-2 \simeq 0.718 \end{cases}$$

$$\left(\phi(x) = x\ln x - x + 1\right)$$

$$\bar\tau = e \Rightarrow \begin{cases} -\frac{1}{T}\ln P_F = \phi(e) = 1 \\ -\frac{1}{T}\ln P_M = \phi(1) = 0 \end{cases}$$