ECE 561: Detection and Estimation Theory
Spring 2009
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- 1)
  Consider the following binary hypothesis testing problem:

  $$\mathrm{H}_0: \quad Y = S; \quad \mathrm{H}_0: \quad Y = S + N; \tag{1}$$

  where S and N are independent random variables with exponential distributions that have parameters a and b, respectively.

  a) Find the Bayesian decision rule and minimum Bayes risk with priors π_0 and π_1, and cost functions

  $$C_{00} = C_{11} = 0, \;\; C_{01} = C_{10} = c > 0. \tag{2}$$

  b) Consider a Neyman-Pearson test without the knowledge of priors and express γ as a function of the probability of false alarm.

  c) Explain how you would derive the minimax test for this scenario. This is a good opportunity to review in more detail the notion of a saddle point and von Neumann's theorem.

- 2)
  Suppose X and Y are random variables. Their joint density is constant in the area $\{(x,y) : x \in [-2,2], y \times [-2,2]\}$, except for

  $$\{(x,y) : \{x \in [-1,0], y \in [-1,1]\} \cup \{x \in [1,2], y \in [1,2]\} \cup \{x \in [1,2], y \in [-2,-1]\}\},$$

  where its value is zero. Let hypothesis H_0 be that the coordinate x satisfies $x \leq 0$, and let H_1 be the hypothesis that $x > 0$. Determine the decision regions for the rule that minimizes the probability of error given a measurement $\mathbf{y} = y$, and determine that probability of error.

  In the (P_D, P_F) plane sketch the receiver operating characteristic (ROC) curve of the likelihood ratio test for this problem. Also, indicate on this plot the region consisting of every (P_D, P_F) value that can be achieved using some decision rule. Is there a point corresponding to $P_D = 2/3$, $P_F = 5/6$? If so, describe a test that achieves this value. If not, explain.

- 3)

  In a binary communication system, messages $m = 0$ and $m = 1$ occur at the output of a source with a priori probabilities 1/4 and 3/4, respectively. Suppose that we observe r , $r = n + m$, where n is a continuous valued random variable with a uniform pdf in $[-3/4, 3/4]$. The random variable n is statistically independent of whether message $m = 0$ or $m = 1$ occurs.

  Find the minimum probability of an error detector, and compute the associated probability of error. Suppose that the receiver does not know the a priori probabilities, so it decides to use a maximum likelihood (ML) detector. Find the ML detector and the associated probability of error. Is the ML detector unique? Justify your answer. If your answer is no, find a different ML receiver and the associated probability of error.

- 4)

  A former 561 student, skilled in the methods of hypothesis testing, came upon a professional gambler who specialized in betting on flips of a coin. Our friend knew that there were three possibilities for the coin the gambler chose:

  a)The coin is fair and the trials are performed independently. That is, each flip of the coin is independent of all other tosses and has an equal probability of coming up heads or tails.
  b) The coin is biased towards heads. Specically, while successive flips of the coin are independent, the probability that any individual flip comes up heads is 3/4.
  c) The successive tosses of the coin are not independent. Specically, while the marginal probability of head (or tails) on any given flip is 1/2, the probability that the next flip yields the same result as the preceding one is only 1/4. For example, the probability that the next flip comes up heads given that the current flip is a head is 1/4. What our friend plans to do is observe two flips of the gamblers coin and then to make a decision among these three possibilities regarding the type of coin the gambler is using.

  Assume that our 561 experts prior assessment is that each of these three possibilities is equally likely and that she views as equally bad any error she might make (i.e. she simply wants to minimize the probability of error). Determine the best decision she can make for each of the possible outcomes of the gamblers pair of coin ips. Determine the probability of error associated with the decision rule determined in the previous part.

- 5)

  Let P be any set of probability distributions on $\mathcal{A}$ and let P_n be the set of those distributions in P which are types of sequences in $\mathcal{A}^n$. Show that for every distribution Q

$$| \frac{1}{n} \log Q^n(\{\mathbf{x} : \hat{P}(\mathbf{x}) \in P\}) + \min_{P \in P_n} D(P||Q) | \leq \frac{\log(n+1)}{n} |\mathcal{A}|, \qquad (3)$$

where $\hat{P}(\mathbf{x})$ denotes the type of the sequence $\mathbf{x}$. Give an intuitive interpretation for this result.