ECE 561: Detection and Estimation Theory
Spring 2009
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- 1)
  Binary frequency shift keying (FSK) on a Rayleigh fading channel can be modeled in terms of a four-dimensional observation vector $\mathbf{Y}$, where $\mathbf{Y}= \mathbf{X}+ \mathbf{Z}$, and where $\mathbf{Z}$ is a zero-mean Gaussian vector, with covariance matrix $\sigma^2 I_{4\times 4}$. Under hypothesis H_0, we have $\mathbf{X}=[X_1\ X_2\ 0\ 0]$, while under hypothesis H_1, we have $\mathbf{X}=[0\ 0\ X_3\ X_4]$. The $\mathbf{X}_i$ are independent, identically-distributed, standard Gaussian random variables. Also, the two hypotheses are equally likely.
  a) Find the maximum likelihood rule for the receiver.
  b) Find the probabilities of false alarm and miss for the maximum likelihood decision rule.

- 2)
  Consider the composite binary hypothesis testing problem for which

  $$f_x(y) = x\ \exp\{-xy\},\ y \geq 0$$

  and

  $$f_x(y) = 0,\ y < 0,$$

  where $x \in [1, \infty)$.

  a) For $\alpha \in (0, 1)$, show that a UMP test of level α exists for testing the hypotheses $H_0 :\ \mathcal{X}_0 = [1, 2)$, and $H_1 :\ \mathcal{X}_1 = [2, \infty)$. Express the test ratio as a function of α.
  b) Assume that the conditional distribution described above is Laplacian instead, with mean $x \in [0, \infty)$ and parameter $\lambda = 1$. Does there exist a UMP test for $H_0 :\ \mathcal{X}_0 = \{0\}$, and $H_1 :\ \mathcal{X}_1 = (0, \infty)$?

- 3)
  Consider the following two forms of generalized maximum likelihood tests for two composite hypotheses:
  a) $\frac{\max_{x\in\mathcal{X}_1}\ f_x(y)}{\max_{x\in\mathcal{X}_0}\ f_x(y)}$;

b) $\frac{E[f_x(y)/x \in \mathcal{X}_1]}{E[f_x(y)/x \in \mathcal{X}_0]}$;
Give two examples for the sets $\mathcal{X}_0$ and $\mathcal{X}_1$ and the conditional distributions, and evaluate the probabilities of false alarm under the two tests, for adequately chosen threshold.

- 4)
  Let f be a convex function. The Legendre transform is defined as $\mathcal{L}(f)(p) = \max_x (p \cdot x - f(x))$.
  a) Find the Legedre transform of $f(x) = x^2$.
  b) Show that the Legendre transform is a convex function of p.
  c) Let $g(p)$ be the Legendre transform of the function f. Then, $d/dp\, g(p) = x(p)$, where $x(p)$ is the solution of the equation $p = d/dx\, f(x)$.
  d) Show that the Legendre transform is an involution.
  e) Two convex functions f and g are said to be dual in the Young sense if one is the Legendre transform of the other. Show that two Young dual functions have to satisfy $p \cdot x \leq f(x) + g(p)$.
  f) Can you find a practical example for the use of Legendre transforms?

- 5)
  Prove Cramer's theorem, which asserts that for a sequence of random variables $\{Z_k,\, k \geq 1\}$,

  $$\lim_{N \to \infty} \frac{1}{N} \log P\{S_N \geq z\} = -\mathcal{L}(G(x))(z)$$

  where $S_N = \frac{1}{N} \sum_{k=1}^{N} Z_k$, and $G(x) = \log E[\exp\{x\, Z\}]$.

- 6)
  Explain how you would use Cramer's theorem in evaluating the asymptotic performance of likelihood tests for which S_N represents a sufficient statistics.