# ECE561: Detection and Estimation Theory

## HW1 Solutions

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### Problem 1

**a)** Bayes decision rule is a likelihood ratio test. The conditional pdf under each hypothesis is:

$$\begin{aligned}
p_0(y) &= a\exp(-ay) \\
p_1(y) &= a\exp(-ax) * b\exp(-bx) \\
p_1(y) &= \frac{ab}{a-b}\left[\exp(-by) - \exp(-ay)\right] \quad , a \neq b \\
p_1(y) &= a^2 y \exp(-ay) \quad , a = b
\end{aligned}$$

where $x, y \geq 0$.

Randomization is not needed here. So, the Bayesian decision rule is:

$$\boxed{\delta_B(y) = \begin{cases} 1 & \text{if } \frac{p_1(y)}{p_0(y)} \geq \frac{\pi_0}{\pi_1} \\ 0 & \text{if } \frac{p_1(y)}{p_0(y)} < \frac{\pi_0}{\pi_1} \end{cases}}$$

where

$$\frac{p_1(y)}{p_0(y)} = \begin{cases} \frac{b}{a-b}\left[\exp(y(a-b)) - 1\right] & , a \neq b \\ ay & , a = b \end{cases}$$

Bayes risk is $\boxed{r(\delta) = \pi_0 R_0(\delta) + \pi_1 R_1(\delta)}$

Let

$\Gamma_1 = \{y : \delta_B(y) = 1\}$

$\Gamma_0 = \{y : \delta_B(y) = 0\}$

Then

$R_0(\delta) = c\int_{y\in\Gamma_1} p_0(y)dy$ and $R_1(\delta) = c\int_{y\in\Gamma_0} p_1(y)dy$

We need to find the minimum Bayes risk for the two possible cases,

**Case 1:** $a \neq b$

$\Gamma_1 = \left\{ y : y \geq \dfrac{1}{a-b} \log \left( \dfrac{a-b}{b} \dfrac{\pi_0}{\pi_1} + 1 \right) = \tau \right\}$

Notice that the threshold ($\tau$) is always positive.

$R_1(\delta) = c \int_0^\tau \frac{ab}{a-b} \left[ \exp\left(-by\right) - \exp\left(-ay\right) \right] dy$

Which results in,

$\boxed{R_1(\delta) = c \left[ 1 - \dfrac{a}{a-b} \exp\left(-b\tau\right) + \dfrac{b}{a-b} \exp\left(-a\tau\right) \right]}$

$\Gamma_0 = \{ y : y < \tau \}$

$\boxed{R_0(\delta) = c \int_\tau^\infty a \exp\left(-ay\right) dy = c \exp(-a\tau)}$

**Case 2:** $a = b$

$\Gamma_1 = \left\{ y : y \geq \frac{\pi_0}{a\pi_1} \right\}$

$\boxed{R_1(\delta) = c \int_0^{\frac{\pi_0}{a\pi_1}} a^2 y \exp\left(-ay\right) dy = c \left[ 1 - \left( \frac{\pi_0}{\pi_1} + 1 \right) \exp\left( -\frac{\pi_0}{\pi_1} \right) \right]}$

$\Gamma_0 = \left\{ y : y < \frac{\pi_0}{a\pi_1} \right\}$

So, $\boxed{R_0(\delta) = c \int_0^{\frac{\pi_0}{b\pi_1}} a \exp\left(-ay\right) dy = c \exp\left( -\frac{\pi_0}{\pi_1} \right)}$

**b)** Neyman-Pearson test takes the form

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } \frac{p_1(y)}{p_0(y)} \geq \gamma \\ 0 & \text{if } \frac{p_1(y)}{p_0(y)} < \gamma \end{cases}$$

For $a \neq b$:

By referring to the analysis in (a), $\tau = \frac{1}{a-b} \log \left( \frac{a-b}{b} \gamma + 1 \right)$

And, $P_F = \exp(-a\tau) = \exp\left( \frac{-a}{a-b} \log \left( \frac{a-b}{b} \gamma + 1 \right) \right)$

So, $\boxed{\gamma = \dfrac{b}{a-b} \left[ \exp\left( -\dfrac{a-b}{a} \log(P_F) \right) - 1 \right] = \dfrac{b}{a-b} \left[ P_F^{\frac{b-a}{a}} - 1 \right]}$

For $a = b$:

By referring to (a), $P_F = \exp(-\gamma)$

So, $\boxed{\gamma = -\log(P_F)}$

**c)** The minimax test corresponds to the intersection between the ROC and the line $P_D = 1 - P_F$. From the analysis in (b),

For $a \neq b$

$\gamma = \dfrac{b}{a-b}\left[\exp\left(-\dfrac{a-b}{a}\log(P_F)\right) - 1\right]$

and $P_D = \displaystyle\int_\gamma^\infty \frac{ab}{a-b}\left[\exp\left(-by\right) - \exp\left(-ay\right)\right]dy = \frac{1}{a-b}\left[a\exp\left(-b\gamma\right) - b\exp\left(-a\gamma\right)\right]$

Substituting the expression we have for $\gamma$ in the previous equation, we get $P_D$ in terms of $P_F$ which generates the ROC curve by letting $P_F$ span [0,1]. Then, the intersection of the ROC with the line $P_D = 1 - P_F$ gives the $(P_F, P_D)$ point for the minimax rule and the slope of the tangent at this point is threshold of the test. For the case of a=b, we follow the same procedure where $\gamma = -\log(P_F)$ and $P_D = \displaystyle\int_\gamma^\infty a^2 y\exp(-ay)dy$.

**Problem 2**

The conditional pdfs for y are:

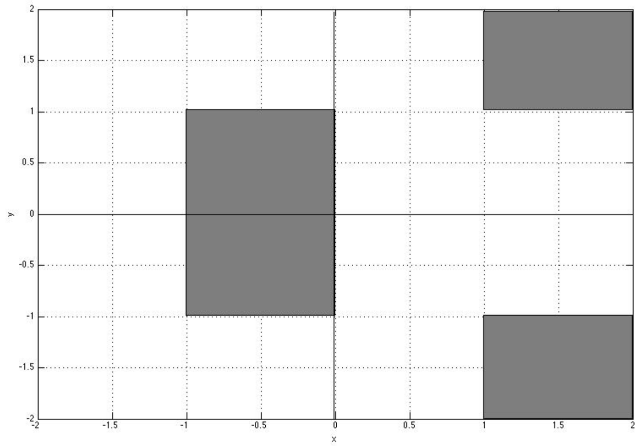
$$p_0(y) = \begin{cases} \frac{1}{3} \quad , & -2 \le y \le -1 \\ \frac{1}{6} \quad , & -1 \le y \le 1 \\ \frac{1}{3} \quad , & 1 \le y \le 2 \end{cases} \qquad\qquad p_1(y) = \begin{cases} \frac{1}{6} \quad , & -2 \le y \le -1 \\ \frac{1}{3} \quad , & -1 \le y \le 1 \\ \frac{1}{6} \quad , & 1 \le y \le 2 \end{cases}$$

So,

$$L(y) = \begin{cases} \frac{1}{2} \quad , & -2 \le y \le -1 \\ 2 \quad , & -1 \le y \le 1 \\ \frac{1}{2} \quad , & 1 \le y \le 2 \end{cases}$$

The region in gray in the figure below shows where the joint pdf of x and y is 0. Since the joint density is constant and the gray area is equal for both hypotheses, $\pi_0 = \pi_1 = 0.5$. So, the threshold for the Bayesian rule = 1 and the optimal decision rule is

$$\boxed{\delta_B(y) = \begin{cases} 1 & \text{if} \quad |y| \le 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}}$$



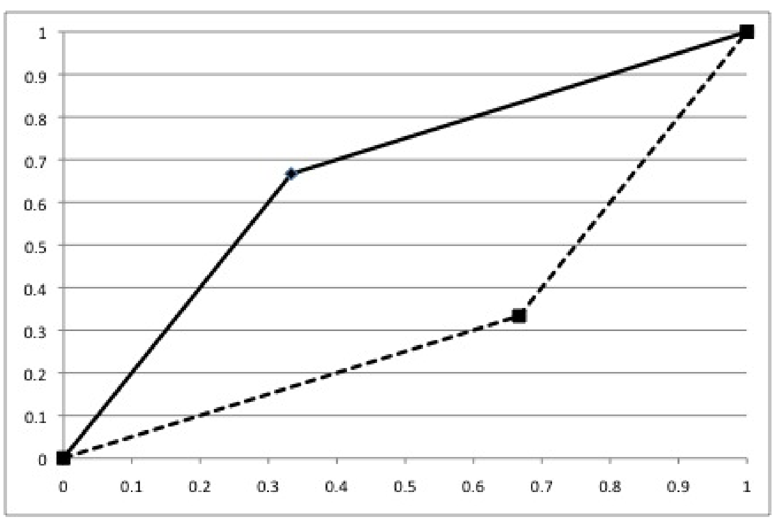
The probability of error is:

$$\boxed{P_E = \frac{1}{2}(\frac{1}{3} + \frac{1}{3}) = \frac{1}{3}}$$

The figure below shows the region of feasible tests. The points on the lines that are not corners are obtained through randomization. Since the likelihood ratio is either $\frac{1}{2}, 2$, or $\infty$, the possible thresholds ($\tau$) that give different $(P_F, P_D)$ are:

$\tau \in (-\infty, \frac{1}{2}], \tau \in (\frac{1}{2}, 2)$ and $\tau \in [2, \infty)$.

The Bayesian decision rule above had a threshold $\tau \in (\frac{1}{2}, 2)$. If we interchange the regions of $H_0$ and $H_1$ we get $(P_F, P_D) = (\frac{2}{3}, \frac{1}{3})$. For $\tau \geq 2$ (always say $H_0$) and $\tau \leq \frac{1}{2}$ (always say $H_1$) we get $(P_F, P_D) = (0, 0)$ and $(P_F, P_D) = (1, 1)$ respectively.



From the discussion above, we see that we couldn't get $(P_F, P_D) = (\frac{5}{6}, \frac{2}{3})$ without randomization. Note that our decision rule should be based on the likelihood ratio. Therefore, we couldn't partition intervals that correspond to a single value of the likelihood ratio. Since we need a high $P_F$, it is intuitive to randomize the rule corresponding to $(P_F, P_D) = (\frac{2}{3}, \frac{1}{3})$ and $(P_F, P_D) = (1, 1)$ to get $P_F \in (\frac{2}{3}, 1)$. Assume that we choose our Bayesian decision rule with probability $\lambda$, and we choose the rule corresponding to $(P_F, P_D) = (1, 1)$ with probability $1 - \lambda$. We want $\frac{5}{6} = \lambda \times \frac{2}{3} + (1 - \lambda) \times 1$. So, $\lambda = \frac{1}{2}$ and $P_D = \frac{1}{2} \times (1 + \frac{1}{3}) = \frac{2}{3}$ and we thus get a rule with $(P_F, P_D) = (\frac{5}{6}, \frac{2}{3})$

**Problem 3**

To find the detector with minimum probability of error, we assume uniform costs and use Bayes rule. The conditional pdfs are:

$p_0(y) = \text{Unif } [\frac{-3}{4}, \frac{3}{4}]$

$p_1(y) = \text{Unif } [\frac{1}{4}, \frac{7}{4}]$

And the likelihood ratio is:

$$L(y) = \frac{p_1(y)}{p_0(y)} = \begin{cases} 0 & , \quad -3/4 \le y \le 1/4 \\ 1 & , \quad 1/4 < y \le 3/4 \\ \infty & , \quad 3/4 < y \le 7/4 \\ \text{undefined} & , \quad \text{otherwise} \end{cases}$$

Randomization is not required since the the likelihood ratio does not take the value of the threshold ($\frac{\pi_0}{\pi_1} = \frac{1}{3}$). The optimal decision rule is:

$$\boxed{\delta_B(y) = \begin{cases} 0 & , \quad -3/4 \le y < 1/4 \\ 1 & , \quad 1/4 \le y \le 7/4 \end{cases}}$$

The error probability is: $\boxed{P_E = \frac{1}{4}\left(\frac{3}{4} - \frac{1}{4}\right) \times \frac{2}{3} + \frac{3}{4}\left(\frac{1}{4} - \frac{1}{4}\right) \times \frac{2}{3} = \frac{1}{12}}$

The ML detector assumes equal priors which gives a threshold = 1. The ML detector in this problem requires randomization and therefore it is not unique. One possible ML detector is:

$$\boxed{\delta_{ML}(y) = \begin{cases} 0 & \text{if} \quad L(y) \leq 1 \\ 1 & \text{if} \quad L(y) > 1 \end{cases}}$$

which is equivalent to:

$$\delta_{ML}(y) = \begin{cases} 0 \ , & -3/4 \leq y < 3/4 \\ 1 \ , & \ \ \ 3/4 \leq y \leq 7/4 \end{cases}$$

$$\boxed{P_E = \frac{1}{2}\left(\frac{3}{4} - \frac{3}{4}\right) \times \frac{2}{3} + \frac{1}{2}\left(\frac{3}{4} - \frac{1}{4}\right) \times \frac{2}{3} = \frac{1}{6}}$$

Another possible ML detector would be:

$$\delta_{ML1}(y) = \begin{cases} 0 & \text{if} \quad L(y) < 1 \\ 1 & \text{if} \quad L(y) \geq 1 \end{cases}$$

This yields $P_E = \frac{1}{2}\left(\frac{3}{4} - \frac{1}{4}\right) \times \frac{2}{3} + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4}\right) \times \frac{2}{3} = \frac{1}{6}$

**Problem 4**

The optimal test in this case is an ML test. Let:

$H_1$: The hypothesis that the coin is fair and the trials are performed independently,

$H_2$: The hypothesis that the coin is biased towards heads with $P_H = 3/4$,

$H_3$: The hypothesis that the successive tosses of the coin are not independent (the probability that the next ip yields the same result as the preceding one is 1/4).

So, $\delta_{ML}(y) = \text{argmax } P(y|H_i)$, where i = {1,2,3}

Consider the four different outcomes, namely {(H,H),(H,T),(T,H),(T,T)}.

**y = (H,H):**

$P(y|H_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(y|H_2) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

$P(y|H_3) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

So, our expert should say $H_2$

**y = (H,T):**

$P(y|H_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(y|H_2) = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$

$P(y|H_3) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

So, our expert should say $H_3$

**y = (T,H):**

$P(y|H_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(y|H_2) = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$

$P(y|H_3) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

So, our expert should say $H_3$

**y = (T,T):**

$P(y|H_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(y|H_2) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

$P(y|H_3) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

So, our expert should say $H_1$

The overall $\boxed{P_E = \dfrac{1-\frac{1}{4}+1-\frac{9}{16}+1-\frac{6}{8}}{3} = \dfrac{23}{48}}$

**Problem 5**

$$\begin{aligned}
Q^n(\{x : \hat{P}(x) \in P\}) &= Q^n(\{x : \hat{P}(x) \in P_n\}) \\
&= \textstyle\sum_{P' \in P_n} Q^n(T(P')) \\
&\leq \textstyle\sum_{P' \in P_n} 2^{-nD(P'||Q)} \\
&\leq \textstyle\sum_{P' \in P_n} 2^{-n \times min_{P' \in P_n} D(P'||Q)} \\
&\leq |P_n| 2^{-n \times min_{P' \in P_n} D(P'||Q)} \\
&\leq (n+1)^{|\mathcal{A}|} 2^{-n \times min_{P' \in P_n} D(P'||Q)}
\end{aligned}$$

So,

$$\frac{1}{n} \log Q^n(\{x : \hat{P}(x) \in P\}) + min_{P' \in P_n} D(P'||Q) \leq \frac{\log(n+1)}{n} |\mathcal{A}|$$

Also,

$$\begin{aligned}
Q^n(\{x : \hat{P}(x) \in P\}) &= Q^n(\{x : \hat{P}(x) \in P_n\}) \\
&= \textstyle\sum_{P' \in P_n} Q^n(T(P')) \\
&\geq Q^n(T(argmin_{P \in P_n} D(P||Q))) \\
&\geq \tfrac{1}{(n+1)^{|\mathcal{A}|}} 2^{-n \times min_{P' \in P_n} D(P'||Q)}
\end{aligned}$$

So,

$\dfrac{1}{n} \log Q^n(\{x : \hat{P}(x) \in P\}) + min_{P' \in P_n} D(P'||Q) \geq -\dfrac{\log(n+1)}{n} |\mathcal{A}|$

Combining the two bounds we get

$|\dfrac{1}{n} \log Q^n(\{x : \hat{P}(x) \in P\}) + min_{P' \in P_n} D(P'||Q)| \leq \dfrac{\log(n+1)}{n} |\mathcal{A}|$

So, given and P,Q; $Q^n(\{x : \hat{P}(x) \in P\})$ decreases exponentially as n $\rightarrow \infty$ and converges to the KL divergence: $min_{P' \in P_n} D(P'||Q)$.