ECE561: Detection and Estimation Theory

HW2 Solutions

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Problem 1 (15 points)

a) The conditional probabilities of the observations under the two hypotheses are:

$$\begin{aligned} p_0(\overline{y}) &= \frac{1}{2\pi(1+\sigma^2)}\exp\left(-\frac{y_1^2+y_2^2}{2(1+\sigma^2)}\right)\cdot\frac{1}{2\pi\sigma^2}\exp\left(-\frac{y_3^2+y_4^2}{2\sigma^2}\right) \\ &= \frac{1}{4\pi^2\sigma^2(1+\sigma^2)}\exp\left(-\frac{y_1^2+y_2^2}{2(1+\sigma^2)}-\frac{y_3^2+y_4^2}{2\sigma^2}\right) \end{aligned}$$

and

$$p_1(\overline{y}) = \frac{1}{4\pi^2\sigma^2(1+\sigma^2)}\exp\left(-\frac{y_1^2+y_2^2}{2(\sigma^2)}-\frac{y_3^2+y_4^2}{2(1+\sigma^2)}\right)$$

So,

$$L(\overline{y}) = \exp\left(-\frac{y_1^2+y_2^2-y_3^2-y_4^2}{2\sigma^2(\sigma^2+1)}\right)$$

and the ML rule is:

$$\boxed{\delta_{ML}(\overline{y}) = \begin{cases} 1 & \text{if} \quad y_3^2+y_4^2 \geq y_1^2+y_2^2 \\ 0 & \text{if} \quad y_3^2+y_4^2 < y_1^2+y_2^2 \end{cases}}$$

b) $P_F = P(y_3^2+y_4^2 \geq y_1^2+y_2^2|H_0)$

Since Y_3, Y_4 are independent Gaussian random variables, $Z_1 = Y_3^2+Y_4^2$ has a chi-squared distribution where

$$p_{Z_1}(z_1) = \frac{1}{2\sigma^2}\exp\left(-\frac{z_1}{2\sigma^2}\right), z_1 \geq 0$$

Similarly, $Z_2 = Y_1^2+Y_2^2$ has a chi-squared distribution where

$$p_{Z_2}(z_2) = \frac{1}{2(\sigma^2+1)}\exp\left(-\frac{z_2}{2(\sigma^2+1)}\right), z_2 \geq 0$$

So,

$$
\begin{aligned}
P_F &= P(Z_1 \geq Z_2 | H_0) \\
&= \int_0^\infty \frac{1}{2(\sigma^2+1)} \exp\left(-\frac{z_2}{2(\sigma^2+1)}\right) dz_2 \int_{z_2}^\infty \frac{1}{2\sigma^2} \exp\left(-\frac{z_1}{2\sigma^2}\right) dz_1 \\
&= \int_0^\infty \frac{1}{2(\sigma^2+1)} \exp\left(-\frac{z_2}{2(\sigma^2+1)}\right) \exp\left(-\frac{z_2}{2\sigma^2}\right) dz_2 \\
&= \frac{1}{2(\sigma^2+1)} \cdot \frac{4\sigma^2(\sigma^2+1)}{2\sigma^2+2(\sigma^2+1)} \\
&= \frac{\sigma^2}{2\sigma^2+1}
\end{aligned}
$$

And,

$$
\boxed{P_M = P(Z_1 < Z_2|H_1) = P(Z_2 > Z_1|H_0) = P_F = \frac{\sigma^2}{2\sigma^2+1}}
$$

Problem 2 (20 points)

a) A UMP test exits if the test threshold is independent of the value of x.

Let $x_1 \in \mathcal{X}_1, x_0 \in \mathcal{X}_0$. Then

$L(y) = \dfrac{p_{x_1}(y)}{p_{x_0}(y)} = \dfrac{x_1}{x_0} \exp((x_0 - x_1)y), y \geq 0$

So, $\delta(y) = \begin{cases} H_1 & \text{if} \quad L(y) \geq \eta \\ H_0 & \text{if} \quad L(y) < \eta \end{cases}$

which could be simplified to

$$
\delta(y) = \begin{cases} H_1 & \text{if} \quad y \leq \dfrac{1}{x_0 - x_1} \log\left(\dfrac{x_0}{x_1}\eta\right) \\ H_0 & \text{if} \quad y > \dfrac{1}{x_0 - x_1} \log\left(\dfrac{x_0}{x_1}\eta\right) \end{cases}
$$

Randomization is not needed because L(y) is continuous. Let $\tau = \dfrac{1}{x_0 - x_1} \log\left(\dfrac{x_0}{x_1}\eta\right)$.

So $P_F = \int_0^\tau x_0 \exp(-x_0 y) dy = 1 - \left(\eta \frac{x_0}{x_1}\right)^{\frac{x_0}{x_1 - x_0}} \leq \alpha$

Since we want $P_F \leq \alpha$ for all x, we observe that P_F is maximized when $x_0 \to 2$. So, $\eta = \dfrac{x_1}{2}(1-\alpha)^{\frac{x_1 - 2}{2}}$

and $\tau = -\dfrac{1}{2}\log(1-\alpha)$.

The threshold (τ) is independent of x. So a UMP exists and the decision rule is:

$$
\boxed{\delta_{UMP}(y) = \begin{cases} H_1 & \text{if} \quad 0 \leq y \leq -\frac{1}{2}\log(1-\alpha) \\ H_0 & \text{if} \quad y > -\frac{1}{2}\log(1-\alpha) \end{cases}}
$$

b) The conditional distributions are:

$p_0(y) = \frac{1}{2}\exp(-|y|)$

$p_1(y) = \frac{1}{2}\exp(-|y-x|)$, $x > 0$

So, $L(y) = \exp(|y| - |y-x|)$, $x > 0$, and

$$\log(L(y)) = \begin{cases} -x & \text{if} \quad y < 0 \\ 2y - x & \text{if} \quad 0 \leq y \leq x \\ x & \text{if} \quad y > x \end{cases}$$

The likelihood ratio is a monotone function of y. Let's check if the test threshold is independent of x.

$$\text{Let, } \delta(y) = \begin{cases} 1 & \text{if} \quad y \geq \tau \\ 0 & \text{if} \quad y < \tau \end{cases}$$

$$P_F = \int_\tau^\infty \frac{1}{2}\exp(-|y|)dy = \begin{cases} \frac{1}{2}\exp(-\tau) & \text{if} \quad \tau \geq 0 \\ 1 - \frac{1}{2}\exp(\tau) & \text{if} \quad \tau < 0 \end{cases}$$

For the UMP test,

$\tau^* = \text{argmax}_\tau P_D(\delta, x)$

If $\tau \geq 0$; $\alpha \leq \frac{1}{2}$ and $\tau = -\log(2\alpha)$, and if $\tau < 0$; $\alpha \geq \frac{1}{2}$ and $\tau = \log(2\alpha - 1)$. So, the test threshold could be expressed independent of x. Thus a UMP test exists and is

$$\boxed{\delta_{UMP}(y) = \begin{cases} 1 & \text{if} \quad y \geq \tau \\ 0 & \text{if} \quad y < \tau \end{cases} \text{ and } \tau = \begin{cases} -\log(2\alpha) & \text{if} \quad 0 \leq \alpha < \frac{1}{2} \\ \log(2(1-\alpha)) & \text{if} \quad \frac{1}{2} < \alpha \leq 1 \end{cases}}$$

Problem 3 (15 points)

Example 1:

Consider the problem:

$H_0 : Y \sim \mathcal{N}(0, 1)$

$H_1 : Y \sim \mathcal{N}(x_1, 1)$, $X_1 \sim \text{Unif}[2, 3]$

Test 1: $\dfrac{\max_{x \in \mathcal{X}_1} f_x(y)}{\max_{x \in \mathcal{X}_0} f_x(y)}$

$$\text{argmax}_{x\in\mathcal{X}_1} f_x(y) = \begin{cases} 2 & \text{if} \quad y < 2 \\ y & \text{if} \quad 2 \leq y \leq 3 \\ 3 & \text{if} \quad y > 3 \end{cases}$$

$$\max_{x\in\mathcal{X}_1} f_x(y) = \begin{cases} \mathcal{N}(2,1) & \text{if} \quad y < 2 \\ 1 & \text{if} \quad 2 \leq y \leq 3 \\ \mathcal{N}(3,1) & \text{if} \quad y > 3 \end{cases}$$

$$\max_{x\in\mathcal{X}_0} f_x(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)$$

$$\log(L(y)) = \begin{cases} 2y-2 & \text{if} \quad y < 2 \\ 0.5y^2 & \text{if} \quad 2 \leq y \leq 3 \\ 3y-4.5 & \text{if} \quad y > 3 \end{cases}$$

Let $\tau = 1$.

$$\begin{aligned} P_F &= P(y<2|H_0)\cdot P(2y-2>1|H_0) + P(2\leq y\leq 3|H_0)\cdot P(0.5y^2>1|H_0) \\ &\quad +P(y>3)\cdot P(3y-4.5>1|H_0) \\ &= 0.0653 + 0.013 + 0 = 0.0783 \end{aligned}$$

Test 2: $\dfrac{\max_{x\in\mathcal{X}_1} E[f_x(y)]}{\max_{x\in\mathcal{X}_0} E[f_x(y)]}$

$\log(L(y)) = 0.5y^2$

Let $\tau = 1$.

$P_F = P(\mathcal{X}^2 > 2) = 0.1573$

Example 2:

Consider the problem:

$H_0 : Y \sim$ Unif$(x_0, 1)$, $X_0 \sim$ Unif$[0.5, 1.5]$

H$_1 : Y \sim$ Unif$(x_1, 1)$, $X_1 \sim$ Unif$[2, 3]$

Test 1:

$L(y) = \frac{1}{4}$

For $\tau = 1$, $P_F = 0$

Test 2:

$L(y) = \frac{1}{2}$

For $\tau = 1,\ P_F = 0$

Problem 4 (20 points)

a) Legendre transform of $f(x) = x^2$:

$\mathcal{L}(f)(p) = \max_x(px - x^2)$

$\nabla\mathcal{L}(f)(p) = 0 \rightarrow p = 2x$

So, $\boxed{\mathcal{L}(f)(p) = p^2/2 - p^2/4 = p^2/4}$

b) Let $g(p) = \mathcal{L}(f)(p)$, and $x(p)$ be the solution to $p = \frac{d}{dx}f(x(p))$. So, $g(p) = px(p) - f(x(p))$.

Then,

$\frac{d}{dp}g(p) = p(\frac{d}{dp}x(p)) + x(p) - \frac{d}{dx}f(x) \cdot \frac{d}{dp}x(p)$

But, $\frac{d}{dx}f(x(p)) = p$.

So, $\frac{d}{dp}g(p) = x(p)$.

$$\begin{aligned}
\frac{d^2}{dp^2}g(p) &= \frac{d}{dp}x(p) \\
&= \frac{d}{dp}x(\frac{d}{dx}f(x(p))) \\
&= \frac{d}{dp}x(\frac{d}{dx}f(x(p))) \cdot \frac{d^2}{dx^2}f(x(p)) \cdot \frac{d}{dp}x(p) \\
&= (\frac{d}{dp}x(p))^2 \cdot \frac{d^2}{dx^2}f(x(p)) \\
&> 0
\end{aligned}$$

So, $g(p) = \mathcal{L}(f)(p)$ is a convex function of p.

c) We know that $g(p) = p \cdot x(p) - f(x(p))$ where $x(p)$ is defined by $p = \frac{d}{dx}f(x(p))$. So
$\frac{d}{dp}g(p) = p \cdot \frac{d}{dp}x(p) + x(p) - \frac{d}{dx}f(x(p)) \cdot \frac{d}{dp}x(p) = x(p)$

d) Need to show that $\mathcal{L}(\mathcal{L}(f))(x) = f(x)$.

Let $g(p) = \mathcal{L}(f)(p)$, and $x(p)$ be the solution to $p = \frac{d}{dx}g(x(p))$.

$\mathcal{L}(\mathcal{L}(f))(x) = \mathcal{L}(g(x)) = xp(x) - g(p(x))$

Now, $g(p)$ is defined as $g(p) = p \cdot y(p) - f(y(p))$ where $y(p)$ is the solution to $p = \frac{d}{dy} f(y(p))$. Also, from (b) $\frac{d}{dp} g(p) = y(p)$. Therefore, we have $p = \frac{d}{dx} g(x(p)) = y(p(x))$

and

$$\begin{aligned} \mathcal{L}(g)(x) &= x \cdot p(x) - g(p(x)) \\ &= y(p(x)) \cdot p(x) - [p(x) \cdot y(p(x)) - f(y(p(x)))] \\ &= f(x) \end{aligned}$$

e) This is essentially by definition

$g(p) = \mathcal{L}(f)(p) = \max_x (p \cdot x - f(x)) \geq p \cdot x - f(x)$ for all p and x. Hence

$g(p) + f(x) \geq p \cdot x$

f) Legendre transform is used to describe the exponent in Chernoff bound and in theory of large deviations.

Problem 5 (15 points)

Refer to Appendix 3.A in the text (P.108-110).

Problem 6 (15 points)

If S_N is the a sufficient statistic, Cramer's theorem gives the rate of decay of miss and false alarm probabilities as the number of observations increases. Let z represent the threshold τ in the test with $-D(p_0||p_1) < \tau < D(p_1||p_0)$. Then,

$P_F = P(S_N \geq \tau | H_0)$ and

$\lim_{N \to \infty} \frac{1}{N} \log P_F = max_u (\tau u - \log E[\exp(u\tau)|H_0])$

The same applies to P_M where $\lim_{N \to \infty} \frac{1}{N} \log P_M = -max_u (\tau u - \log E[\exp(u\tau)|H_1])$