ECE561: Detection and Estimation Theory

HW3 Solutions

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Problem 1 (20 points)

a) The Bayesian decision rule with equal priors is an ML rule. So

$\delta_{ML}(y) = \text{argmax}_j p_j(y),\ j = 0, 1, ..., 4$

where $p_j(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y - j + 2)^2\right),\ j = 0, 1, ..., 4$ So, $\delta_{ML}(y) = \text{argmin}_j (y - j + 2)^2$ which results in

$$\delta_{ML}(y) = \begin{cases} 0 & \text{if} & & & y & \leq & -1.5 \\ 1 & \text{if} & -1.5 & < & y & \leq & -0.5 \\ 2 & \text{if} & -0.5 & < & y & \leq & 0.5 \\ 3 & \text{if} & 0.5 & < & y & \leq & 1.5 \\ 4 & \text{if} & & & y & > & 1.5 \end{cases}$$

b) $r(\delta_{ML}) = 0.2 \sum_{j=0}^{4} P(error|H_j)$

So, $\boxed{r(\delta) = 0.2(2 \times Q(0.5) + 3 \times 2 \times Q(0.5)) = 1.6 \times Q(0.5)}$

Problem 2 (20 points)

Since π_0 is independent of the observations,

$E_{\pi_0}[\pi_0 R_0(\delta) + (1 - \pi_0) R_1(\delta)] = E[\pi_0] R_0(\delta) + (1 - E[\pi_0]) R_1(\delta)$

The optimal rule the optimal Bayesian test with $E[\pi_0]$ replacing π_0. So, the optimal rule is

$$\delta(y) = \begin{cases} 1 & \text{if} \quad L(y) \geq \tau \\ 0 & \text{if} \quad L(y) < \tau \end{cases}$$

where, $\tau = \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \cdot \frac{E[\pi_0]}{1 - E[\pi_0]}$

Problem 3 (20 points)

a) The likelihood ratio is:

$$L_N(\overline{y}) = \begin{cases} \dfrac{w_0}{w_1} & \text{if} \quad |y_i| \leq \dfrac{w_0}{2} \text{ for all } i \in \{1, .., N\} \\ \infty & \text{if} \quad |y_i| \in \left(\dfrac{w_0}{2}, \dfrac{w_1}{2}\right] \text{ for at least one } i \in \{1, ..., N\} \end{cases}$$

So, an additional sample is drawn if:

- $|y_i| \leq \frac{w_0}{2}$, and
- Number of samples with $|y_i| \leq \frac{w_0}{2}$ is less than $\eta = \left\lceil \dfrac{\log(A)}{\log(w_0) - \log(w_1)} \right\rceil$

b) $P_F = P(L_N(\overline{y}) \geq B|H_0) = 0$ because when H_0 is true, $L_N(\overline{y}) < 1$ and we know that $B > 1$.

$$P_M = P\left(|y| \leq \frac{w_0}{2}|H_1\right)^\eta = \left(\frac{w_0}{w_1}\right)^\eta$$

c) Clearly, $E[N|H_0] = \eta$

$$\begin{aligned} E[N|H_1] &= \left(1 - \frac{w_0}{w_1}\right) + 2 \cdot \left(1 - \frac{w_0}{w_1}\right)\frac{w_0}{w_1} + 3 \cdot \left(1 - \frac{w_0}{w_1}\right)\left(\frac{w_0}{w_1}\right)^2 + ... \\ &\quad + (\eta - 1) \cdot \left(1 - \frac{w_0}{w_1}\right)\left(\frac{w_0}{w_1}\right)^{(\eta-2)} + \eta \cdot \left(1 - \frac{w_0}{w_1}\right)\left(\frac{w_0}{w_1}\right)^{(\eta-1)} + \eta \cdot \left(\frac{w_0}{w_1}\right)^\eta \\ &= \frac{1 - \left(\frac{w_0}{w_1}\right)^\eta}{1 - \left(\frac{w_0}{w_1}\right)} \end{aligned}$$

Problem 4 (20 points)

a) $L_N(\overline{y}) = \left(\dfrac{p_1}{p_0}\right)^{N_1(n)} \cdot \left(\dfrac{q_1}{q_0}\right)^{(n-N_1(n))}$

So,

$$\begin{aligned} \log(L_N(\overline{y})) &= N_1(n) \cdot \log\left(\frac{p_1}{p_0}\right) + (n - N_1(n)) \cdot \log\left(\frac{q_1}{q_0}\right) \\ &= N_1(n) \cdot \left(\log\left(\frac{p_1}{p_0}\right) - \log\left(\frac{q_1}{q_0}\right)\right) + n\log\left(\frac{q_1}{q_0}\right) \end{aligned}$$

The optimal Bayesian rule for the SPRT is

$$\delta(\overline{y}) = \begin{cases} 1 & \text{if} \quad \log(L_n(\overline{y})) \geq \log(B) \\ 0 & \text{if} \quad \log(L_n(\overline{y})) \leq \log(A) \\ \text{take another sample} & \text{if} \quad \text{otherwise} \end{cases}$$

By substituting the expression we have for $\log(L_n(\overline{y}))$ we get

$$\delta(\overline{y}) = \begin{cases} 1 & \text{if} \quad N_1(n) \geq \dfrac{\log(B) - n \cdot \log(q_1/q_0)}{\log(p_1/p_0) - \log(q_1/q_0)} \\ 0 & \text{if} \quad N_1(n) \leq \dfrac{\log(A) - n \cdot \log(q_1/q_0)}{\log(p_1/p_0) - \log(q_1/q_0)} \\ \text{take another sample} & \text{if} \quad \text{otherwise} \end{cases}$$

So, we take another sample if $c + \eta n < N_1(n) < d + \eta n$, where

$$c = \frac{a}{\log(p_1/p_0) - \log(q_1/q_0)}$$

$$d = \frac{b}{\log(p_1/p_0) - \log(q_1/q_0)}$$

$$\eta = \frac{\log(q_0/q_1)}{\log(p_1/p_0) - \log(q_1/q_0)}$$

b) $a = \log(A) = \log\left(\frac{P_M}{1-P_F}\right) = \log\left(\frac{10^{-4}}{1-10^{-4}}\right) \approx -9.2$

$b = \log(B) = \log\left(\frac{1-P_M}{P_F}\right) = \log\left(\frac{1-10^{-4}}{10^{-4}}\right) = -a \approx 9.2$

So, $\boxed{c = -3.32,\ d = 3.32,\ \eta = 0.5}$

c) $D(P_0||P_1) = \log(p_0/p_1) \cdot p_0 + \log(q_0/q_1) \cdot q_0 = \frac{6}{5}\log(2)$

$D(P_1||P_0) = \log(p_1/p_0) \cdot p_1 + \log(q_1/q_0) \cdot q_1 = \frac{6}{5}\log(2)$

$E[N|H_0] = \dfrac{1}{-D(P_0||P_1)} \cdot \left[P_F \cdot \log\left(\dfrac{1-P_M}{P_F}\right) + (1-P_F)\log\left(\dfrac{P_M}{1-P_F}\right)\right] = 11.07$

$E[N|H_1] = 11.07$

d) $C(P_0, P_1) = -\min_{0 \leq u \leq 1} \log\left(p_1^u p_0^{(1-u)} + (1-p_1)^u (1-p_0)^{(1-u)}\right)$

$$\begin{aligned} u^* &= \text{argmin}_{0 \leq u \leq 1} \log\left(p_1^u p_0^{(1-u)} + (1-p_1)^u (1-p_0)^{(1-u)}\right) \\ &= 0.5 \end{aligned}$$

$C(P_0, P_1) = -\log(2 \cdot \sqrt{p_1 \cdot p_0}) = \log(1.25)$

For $P_F = P_M = P_E$
$N \approx \frac{-\log(P_E)}{C(P_0, P_1)} = 41.27$
So, the reduction in the number of samples when using SPRT is $\sim 73\%$.

Problem 5 (20 points)

a) From the observation mentioned in the question, the optimal test for at most $k+1$ collected observations should be optimal for all tests that have observations up to $k+1$, i.e $k, k-1, ..., 1$ observations. Therefore, the following should also be satisfied:
$V_{k+1}(\pi_0) = \min\{T(\pi_0), D + E_{Y_1}[V(\pi_0(Y_1))]\}$
Since we didn't go with the choice of deciding without any observations, $T(\pi_0) > D + E_{Y_1}[V(\pi_0(Y_1))]$. After each observation, the cost of the decision increases (getting closer to $T(\pi_0)$) and at k+1 the cost of the decision is:
$V_{k+1}(\pi_0) = \min\{T(\pi_0), D + E_{Y_1}[V_k(\pi_0(Y_1))]\}$

b) $T(\pi_0) = \min\{C_F\pi_0.C_M(1-\pi_0)\} = \begin{cases} \pi_0 & \text{if} \quad \pi_0 \leq 0.5 \\ 1-\pi_0 & \text{if} \quad \pi_0 > 0.5 \end{cases}$
And, $\pi_0(Y_1) = \dfrac{\pi_0 p_0(Y_1)}{\pi_0 p_0(Y_1) + (1-\pi_0)p_1(Y_1)}$

$$\begin{aligned} E_{Y_1}[V_0(\pi(Y_1))] &= P(Y_1 = 1)\cdot V_0(\pi(1)) + P(Y_1 = 0)\cdot V_0(\pi(0)) \\ &= \begin{cases} \pi_0 & \text{if} \quad 0 \leq \pi_0 \leq 0.2 \\ \pi_0 & \text{if} \quad 0.2 < \pi_0 \leq 0.8 \\ 1-\pi_0 & \text{if} \quad 0.8 < \pi_0 \leq 1 \end{cases} \end{aligned}$$

So, $\mathrm{V}_1(\pi_0) = \min\{T(\pi_0), D + E_{Y_1}[V_0(\pi(Y_1))]\} = \begin{cases} \pi_0 & \text{if} \quad 0 \leq \pi_0 \leq 0.3 \\ 0.3 & \text{if} \quad 0.3 \leq \pi_0 \leq 0.7 \\ 1-\pi_0 & \text{if} \quad 0.7 \leq \pi_0 \leq 1 \end{cases}$

Thus, $\pi_L^{(1)} = 0.3, \pi_U^{(1)} = 0.7$

Iterate...

$$V(\pi_0) = \begin{cases} \pi_0 & \text{if} \quad 0 \leq \pi_0 \leq 0.258 \\ 0.26\pi_0 + 0.19 & \text{if} \quad 0.258 \leq \pi_0 \leq 0.418 \\ 0.3 & \text{if} \quad 0.418 \leq \pi_0 \leq 0.582 \\ -0.04\pi_0 + 0.32 & \text{if} \quad 0.582 \leq \pi_0 \leq 0.742 \\ -0.93\pi_0 + 0.93 & \text{if} \quad 0.742 \leq \pi_0 \leq 1 \end{cases}$$

And,

$$\boxed{\pi_L = 0.258, \pi_I = 0.742}$$