ECE561: Detection and Estimation Theory

HW4 Solutions

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Problem 1 (20 points)

Refer to Levy, p250-263

Problem 2 (20 points)

The MMSE, MAE, MAP estimators are associated with the mean, median, and mode respectively. It was shown in class that if the conditional density is symmetric around the mode, the loss function is even and convex, then the three estimators are equal. However, if the distribution is skewed, then the estimators will not be equal. If the distribution is right-skewed, then

MMSE>MAE>MAP

If the distribution is left-skewed, then the reverse is true, i.e., MAP>MAE>MMSE.

Problem 3 (20 points)

Consider the estimator for the general case

$\tilde{X} = \tilde{A}\tilde{Y} + \tilde{b}$

where $\tilde{A}$ is a $m \times n$ matrix with full rank.

Let $\tilde{K} = \tilde{A}^T\tilde{A}$. Then, $\tilde{K}$ is a symmetric positive-definite matrix. Using Cholesky decomposition, $\tilde{K}$ could be decomposed into a product of a lower and upper triangular matrices, .i.e., $\tilde{K} = LL^T$ where L is a $n \times n$ lower triangular matrix. Note that since $\tilde{K}$ is symmetric, $\tilde{K} = \tilde{K}^T$ and so $\tilde{K}$ could also be expressed as $\tilde{K} = UU^T$ where $U = L^T$ is an upper triangular matrix.

$$\begin{aligned}\tilde{A}^T\tilde{X} &= \tilde{K}\tilde{Y} + \tilde{A}^T\tilde{b} \\ &= LL^T\tilde{Y} + \tilde{A}^T\tilde{b}\end{aligned}$$

Let, $\hat{X} = \tilde{A}^T\tilde{X}$, $Y = L^T\tilde{Y}$, $b = \tilde{A}^T\tilde{b}$, $A = L$. Then,

$\hat{X} = AY + b$, where A is a triangular matrix. So, $\hat{X} = \tilde{A}^T\tilde{X}$ is the desired estimator with A restricted to a triangular matrix.

When does the LLS estimator allow for A to be triangular?

$A = K_{XY}K_Y^{-1}$ is triangular if K_{XY} and K_Y^{-1} are triangular. If the observations are Y are uncorrelated, then K_Y is a diagonal matrix which is also triangular. K_{XY} will also be triangular of X and Y are uncorrelated.

Problem 4 (20 points)

a) $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Let $G = \begin{bmatrix} I & 0 \\ -D^{-1} & I \end{bmatrix}$

Then, $MG = \begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix}$

Since $\det(G) = 1$, we have $\det(MG) = \det(M)$.

$$\begin{aligned}\det(MG) &= \det([A - BD^{-1}C]D) \\ &= \det(A - BD^{-1}C) \times \det(D) \\ &= \det(D/M) \times \det(D)\end{aligned}$$

So, $\det(M) = \det(D) \times \det(D/M)$

b) Let $G = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix}$

Then,

$$GMG^T = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -(D^{-1})^T B^T & I \end{bmatrix}$$

$$= \begin{bmatrix} A - BD^{-1}C & 0 \\ C - D(D^{-1})^T B^T & D \end{bmatrix}$$

Since M is Hermitian, $B^T = C$. So,

$$GMG^T = \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}$$

$$= \begin{bmatrix} D/M & 0 \\ 0 & D \end{bmatrix}$$

By Sylvester's law of inertia, if A and B are $n \times n$ Hermitian matrices, then there exists a non-singular $n \times n$ matrix G such that $B = GAG^T$ and $I_A = I_B$.

So, $I_{GMG^T} = I_M + I_D + I_M$

c) The LLSE error-covariance matrix is the equal to the Schur complement. The Schur complement is

$S = K_X - K_{XY} K_Y^{-1} K_{YX}$.

Intuitively, if we do not have any observations, the MMSE estimator will be $E[X]$, hence the K_X term. However, as we get observations, the error will decrease.

Problem 5 (20 points)

a) Let $V = \frac{\partial}{\partial x} \log f_{X|Y}(x|y)$

Based on Hölder's inequality:

$E[|\hat{X} - E[\hat{X}]|^3] \geq \frac{1}{E[|V - E[V]|^{3/2}]^2}$

b) Let

h = number of heads appearing in n flips of a coin,

t = number of tails appearing in n flips of a coin,

p: probability that a head appears.

Since h is a sufficient statistic, the Fisher information is:

$$
\begin{aligned}
J(h) &= -E\left[\frac{\partial^2}{\partial h^2}\log p(h|p)\right] \\
&= -E\left[\frac{\partial^2}{\partial h^2}\log\left(p^h(1-p)^t\frac{n!}{h!t!}\right)\right] \\
&= -E\left[\frac{\partial^2}{\partial h^2}\left(h\log(p)+t\log(1-p)\right)\right] \\
&= -E\left[\frac{h}{p^2}+\frac{t}{(1-p)^2}\right] \\
&= \frac{n}{p}+\frac{n}{1-p} \\
&= \frac{n}{p(1-p)}
\end{aligned}
$$