ECE563 Fall 2004 **Homework 10**

Date Assigned: 27 October 2004.
Date Due: 3 November 2004 in class.
Suggested Reading: Sections 10.1 through 10.3 of your text book.

1. Show that among the class of non-negative random variables with mean μ, the *exponential* random variable with parameter μ, i.e., it has the density function

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}, \qquad x \geq 0,$$

has the largest differential entropy.

2. Problems 10.1 and 10.2 from your text.

3. A memoryless channel has an input X contrained to the interval $(-0.5, 0.5)$, and has additive noise Z with the probability density

$$f(z) = \begin{cases} 0.5 & \text{for } -1 < z \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $Y = X + Z$. Find the capacity of the channel and the input distribution that leads to it.
Hint: It is easier to guess the input distribution and then verify that it achieves the capacity.

4. Consider a discrete memoryless channel $(\mathcal{X}, p(y|x), \mathcal{Y})$. Suppose that there is a *cost* $b(x)$ associated with each input symbol $x \in \mathcal{X}$. As usual, an $(n, 2^{nR})$ block code of length n and rate R is given by encoding and decoding maps

$$e_n : \{1, \ldots, 2^{nR}\} \mapsto \mathcal{X}^n$$
$$d_n : \mathcal{Y}^n \mapsto \{1, \ldots, 2^{nR}\},$$

and the average error probability of the code is defined as

$$p_e^{(n)} = \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \mathbb{P}\left(d_n(e_n(w)) \neq w\right).$$

The *average cost* of a codeword w is given by

$$b(w) = \frac{1}{n} \sum_{k=1}^{n} b(x_k(w)),$$

where $e_n(w) = (x_1(w), \ldots, x_n(w))$. We say that reliable communication is possible over the channel at rate R and at cost β if there is a sequence of $(n, 2^{nR})$ codes such that $\limsup_{n\to\infty} p_e^{(n)} = 0$, and

$$\frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} c(w) \leq \beta, \qquad \forall n.$$

(a) Sketch an argument (you only need to show the exact steps where the proof differs from the situation when there is no cost) to show that for any R less than the *information capacity at cost* β, given by

$$C(\beta) = \max_{p(x):\sum_{x\in\mathcal{X}} p(x)b(x)\leq\beta} I(X;Y),$$

reliable communication is possible over the channel at cost β.
Hint: You might need to restrict the set of "typical sequences" $A_\epsilon^{(n)}$ used in the proofs in Chapter 8. Though you dont really need that level of generality for this problem, it might help to take a look at pages 370-371 of your text book.

(b) Prove (rigorously) that communication is not possible over the channel at cost β at rates that exceed $C(\beta)$.
Hint: This part is relatively easy. It might help to first prove that $C(\beta)$ is a concave-$\cap$ function of β.

(c) How does the capacity-cost function of a discrete memoryless channel relate to its Shannon capacity?