ECE563 Fall 2004 **Homework 11**

Date Assigned: 3 November 2004.
Date Due: 10 November 2004 in class.
Suggested Reading: Nothing in particular; this exercise tests your understanding of the basic discrete memoryless channel setting. So, it might help to really understand the forward and converse proofs of the main capacity theorem (Sections 8.7 and 8.9 of your text). You can learn more about the geometric proof of the capacity of the AWGN channel, as discussed in lecture, from Appendix B.5 in [2].

1. *Sum Channel*
Consider K discrete memoryless channels with respective input and output alphabets $\mathcal{X}_k$ and $\mathcal{Y}_k$ and with respective information capacity C_k, $1 \leq k \leq K$. Suppose that both the input and output alphabets of the different channels are disjoint. The *sum channel* associated to these channels is the channel whose input alphabet is the *union*

$$\mathcal{X} := \bigcup_{k=1}^{K} \mathcal{X}_k,$$

of the individual input alphabets, the output alphabet is the union

$$\mathcal{Y} := \bigcup_{k=1}^{K} \mathcal{Y}_k,$$

of the respective output alphabets, and such that if input letter $x \in \mathcal{X}_k$ is used the output is $y \in \mathcal{Y}_k$ with the channel transition probabilities corresponding to channel k. In other words, at each time the transmitter may choose to use any one of the K channels and transmit any symbol from the input alphabet of this channel - if the transmitter chooses a letter from channel k the receiver will see a symbol from the output alphabet of channel k at that time.

   (a) Show that the capacity of the sum channel is given by

$$C = \log\left(\sum_{k=1}^{K} 2^{C_k}\right), \tag{1}$$

   and find an expression for the optimal input probability distribution for the sum channel in terms of the optimal input probability distributions for the individual channels.

   (b) Interpret the expression for C in (1) as the average of the capacities of the individual channels plus the information conveyed by the *selection* of the channel.

2. *Parallel Channel*
A set of K discrete memoryless channels have the capacities $C_1, \ldots, C_K$. Let these channels be *connected in parallel* in the sense that each unit of time an arbitrary symbol is transmitted and received over *each* channel. Thus the input $\mathbf{x} = (x_1, \ldots, x_K)$ to the parallel channel is a K-tuple the components of which are inputs to the individual channels, and the output $\mathbf{y} = (y_1, \ldots, y_K)$ is a K-tuple whose components are the individual channel outputs. Prove that the capacity of the parallel channel is

$$\sum_{k=1}^{K} C_k.$$

Assume that the output from each channel is statistically related *only* to the input of that channel, i.e.,

$$\mathbb{P}\left[\mathbf{y}|\mathbf{x}\right] = \Pi_k \mathbb{P}\left[y_k|x_k\right].$$

Hint: See Section 10.4; in particular, you might want to carefully go over the inequalities in the upper part of page 252 of your text.

3. *Compound Channel*
Consider a set of K channels with the same input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. One of these K channels is arbitrarily picked and revealed only to the receiver. The transmitter is ignorant of the channel picked but still has to communicate reliably over the channel, whichever is picked. Such a channel is called a *compound channel*. Show that the capacity of the compound channel is

$$C = \max_{p(x)} \min_{k=1\ldots K} I_k(X;Y), \tag{2}$$

where we have denoted the mutual information between an input X and the output of the k^{th} channel by $I_k(X;Y)$.
Hint: The forward part is very straightforward; simply state why the capacity is at least as large as the RHS of (2). The converse is only a little more complicated than that of the plain DMC. You can follow most of the steps in Section 8.9 except that you might want to keep in mind that $I_k(X;Y)$ is *concave* in the input distribution $p(x)$.
Commentary: Somewhat remarkably, the capacity of the compound channel remains unchanged even when the the channel picked is not revealed to the receiver. You can learn more about compound channels and *arbitrarily varying* channels (where the person picking the channel actually knows the codebook and can thus act nefariously) from [1].

References

[1] A. Lapidoth and P. Narayan, "Reliable communication under channel uncertainty", *IEEE Transactions on Information Theory*, Vol. 44(6), pp. 2148-2177, October 1998.

[2] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005. A draft of this upcoming book can be downloaded from `www.ifp.uiuc.edu/~pramodv/pubs/book090904.pdf` .