ECE 563 Homework Due: October 22, 2009

Read pages 65-80 of Shannon and Weaver

1. Prove that for any discrete random variables X, Y, and Z,

$$I(X;(Y,Z)) \quad = \quad I(X;Z|Y) + I(X;Y).$$

Suppose that the probability distribution of Y, conditional on X and Z, is the same as the probability distribution of Y, conditional on Z. That is,

$$P_{k|j\ell} \quad = \quad P_{k|j}$$

for all k, j, ℓ. Prove that

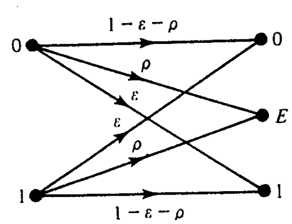
$$I(Z;Y|X) \quad = \quad 0.$$

2. The channel with probability transition matrix

$$\boldsymbol{Q} \quad = \quad \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

exhibits a kind of symmetry. Does it satisfy our definition of a symmetric channel? What probability distribution achieves the capacity?

3. The binary errors-and-erasures channel is given by the following illustration



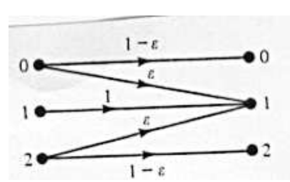
a. Find the capacity.

b. Specialize to erasures only ($\epsilon = 0$).

c. Specialize to the binary symmetric channel ($\rho = 0$).

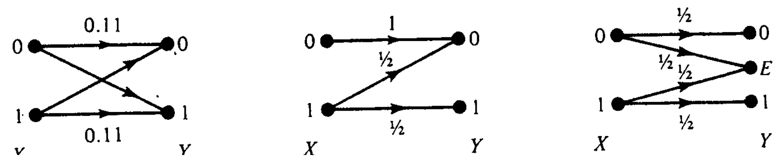
d. Would you prefer a binary symmetric channel with crossover probability = 0.125 or a binary erasure channel with probability of erasure = 0.5?

4. Consider the following channel:



Is $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ an input distribution that achieves capacity?

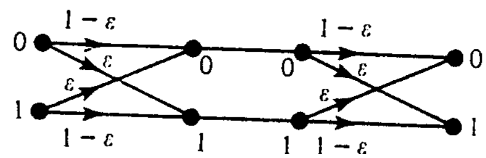
5. Each of the three binary channels shown below is used with the probability distribution on the input letters of $\boldsymbol{p} = (p_0, p_1) = \left(\frac{1}{2}, \frac{1}{2}\right)$. In each case find $I(X;Y)$.



Which channel, with this distribution $\boldsymbol{p}$ on the input letters, has the largest mutual information between input and output? For which channel, if any, does this $\boldsymbol{p}$ not achieve capacity?

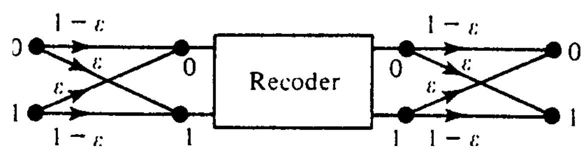
6. Suppose that two channels are to be connected end to end to form a composite channel. Each of the original channels is a binary symmetric channel with the crossover probability ϵ.

a. Suppose that the output of the first channel is connected directly to the input of the second with no processing between as shown.



What is the capacity of the composite channel?

b. Suppose that a decoder/encoder (a recoder) is allowed between the channels as shown:



What is the capacity?

7. A binary symmetric Markov channel has a binary input alphabet and probability transition matrix $[P_{j|i}]$, given by

$$\boldsymbol{P} = \begin{bmatrix} 1-\rho & \rho \\ \rho & 1-\rho \end{bmatrix}.$$

Find the capacity.

8. Prove the following statement, which is known as the *convex decomposition lemma*:

$$\log \sum_j p_j Q_{k|j} = \max_{\boldsymbol{P}} \sum_j P_{j|k} \log \frac{p_j Q_{k|j}}{P_{j|k}}.$$

The convex decomposition lemma replaces the logarithm of a sum — which can be clumsy to deal with — by the sum of logarithms, which can be easy to deal with, but at the cost of a gratuitous maximization.