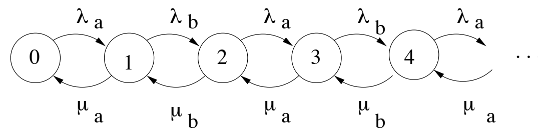
**ECE 467 COMMUNICATION NETWORK ANALYSIS** **FALL 2006**
**Solutions to Exam 1**

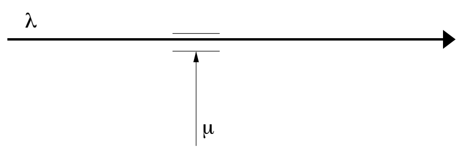
1. Consider a single server queueing system in which the number in the system is modeled as a continuous time birth-death process with the transition rate diagram shown, where $\lambda_a, \lambda_b, \mu_a$, and $\mu_b$ are strictly positive constants.




(a) Under what additional assumptions on these four parameters is the process positive recurrent?
(b) Assuming the system is positive recurrent, under what conditions on $\lambda_a, \lambda_b, \mu_a$, and $\mu_b$ is it true that the distribution of the number in the system at the time of a typical arrival is the same as the equilibrium distribution of the number in the system?

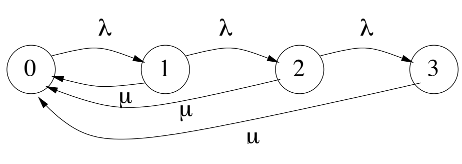
SOLUTION (a) Let $\theta = \frac{\lambda_a \lambda_b}{\mu_a \mu_b}$. Then $S_1 = \sum_{n=0}^{\infty} \theta^n (1 + \frac{\lambda_a}{\mu_a})$, which is finite if and only if $\theta < 1$. Thus, the process is positive recurrent if and only if $\theta < 1$. (In case $\theta < 1$, $\pi_{2n} = \theta^n / S_1$ and $\pi_{2n+1} = \theta^n \frac{\lambda_a}{S_1 \mu_a}$.)
(b) $\lambda_a = \lambda_b$. In general, $r_k = \pi_k \lambda_k / \overline{\lambda}$, so that $r \equiv \pi$ if and only if the arrival rates are all equal (corresponding to Poission arrivals, and then PASTA holds.)

2. Suppose that a regulated communication link resets at a sequence of times forming a Poisson process with rate $\mu$. Packets are offered to the link according to a Poisson process with rate $\lambda$. Suppose the link shuts down after three packets pass in the absence of resets. Once the link is shut down, additional offered packets are dropped, until the link is reset again, at which time the process begins anew.



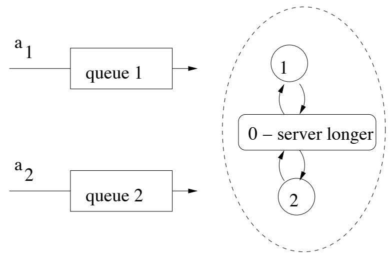

(a) Sketch a transition rate diagram for a finite state Markov process describing the system state.
(b) Express the dropping probability (same as the long term fraction of packets dropped) in terms of $\lambda$ and $\mu$.

SOLUTION (a) Let $\mathcal{S} = \{/, \infty, \in, \ni\}$, where the state is the number of packets passed since the last reset.




(b) By the PASTA property, the dropping probability is $\pi_3$. We can find the equilibrium distribution $\pi$ by solving the equation $\pi Q = 0$. The balance equation for state 0 is $\lambda \pi_0 = \mu(1 - \pi_0)$ so that $\pi_0 = \frac{\mu}{\lambda+\mu}$. The balance equation for state $i \in \{1, 2\}$ is $\lambda \pi_{i-1} = (\lambda + \mu)\pi_i$, so that $\pi_1 = \pi_0(\frac{\lambda}{\lambda+\mu})$ and $\pi_2 = \pi_0(\frac{\lambda}{\lambda+\mu})^2$. Finally, $\lambda \pi_2 = \mu \pi_3$ so that $\pi_3 = \pi_0(\frac{\lambda}{\lambda+\mu})^2 \frac{\lambda}{\mu} = \frac{\lambda^3}{(\lambda+\mu)^3}$. The dropping probability is $\pi_3 = \frac{\lambda^3}{(\lambda+\mu)^3}$. (This formula for $\pi_3$ can be deduced with virtually no calculation from the properties of merged Poisson processes. Fix a time $t$. Each event is a packet arrival with probability $\frac{\lambda}{\lambda+\mu}$ and is a reset otherwise. The types of different events are independent. Finally, $\pi_3(t)$ is the probability that the last three events before time $t$ were arrivals. The formula follows.)

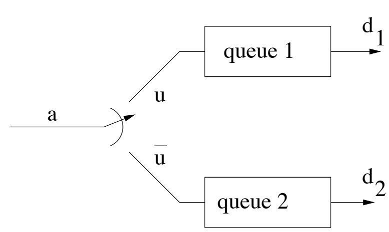
3. Consider two queues, queue 1 and queue 2, such that in each time slot, queue $i$ receives a new packet with probability $a_i$, where $0 < a_1 < 1$ and $0 < a_2 < 1$. Suppose the server is described by a three state Markov process, as shown.




If the server process is in state $i$ for $i \in \{1, 2\}$ at the beginning of a slot, then a potential service is given to station $i$. If the server process is in state 0 at the beginning of a slot, then a potential service is given to the longer queue (with ties broken in favor of queue 1). Then during the slot, the server state jumps with the probabilities indicated. (Note that a packet can arrive and depart in one time slot.) For what values of $a_1$ and $a_2$ is the process stable? Briefly explain your answer (but rigorous proof is not required).

SOLUTION By inspection, the equilibrium distribution of the server state is $(w(0), w(1), w(2)) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Thus, over the long run, the server can serve queue 1 at most $\frac{3}{4}$ of the time, it can serve queue 2 at most $\frac{3}{4}$ of the time, and, of course, it can offer at most one service in each slot. Thus, it is necessary that $a_1 < \frac{3}{4}, a_2 < \frac{3}{4}$, and $a_1 + a_2 < 1$. (It can be shown that these conditions are also sufficient for stability by using the Lyapunov function $V(x) = \frac{x_1^2+x_2^2}{2}$ applied to the time-sampled process $Y(n) = X(nk)$, where $k$ is a constant large enough so that the mean fraction of time the server is in each state over $k$ time slots is close to the statistical average, no matter what the state of the server at the beginning of such interval.)

4. Consider the system of Example 1.b in the notes (a discrete time model, using the route to shorter policy, with ties broken in favor of queue 1, so $u = I_{\{x_1 \le x_2\}}$):




Assume $a = 0.7$ and $d_1 = d_2 = 0.4$. The system is positive recurrent. Explain why the function $V(x) = x_1 + x_2$ does *not* satisfy the Foster-Lyapunov stability criteria for positive recurrence, for any choice of the constant $b$ and the finite set $C$.

SOLUTION Suppose $x$ is on the postive $x_2$ axis (i.e. $x_1 = 0$ and $x_2 > 0$). Then, given $X(t) = x$, during the slot, queue 1 will increase to 1 with probability $a(1 - d_1) = 0.42$, and otherwise stay at zero. Queue 2 will decrease by one with probability 0.4, and otherwise stay the same. Thus, the drift of $V$, $E[V(X(t + 1) - V(x)|X(t) = x]$ is equal to 0.02. Therefore, the drift is strictly positive for infinitely many states, whereas the Foster-Lyapunov condition requires that the drift be negative off of a finite set $C$. So, the linear choice for $V$ does not work for this example.