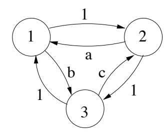
Solutions to ECE 567 Exam 2 **Fall 2006**

1. (5 points) Consider the continuous-time Markov process with the transition rate diagram shown. Identify all possible values of $(a, b, c) \in \mathbb{R}^3_+$ which make the process reversible.

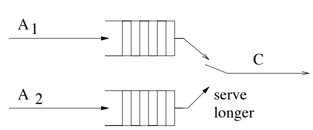


SOLUTION
The process is reversible and $\pi = (\pi_1, \pi_2, \pi_3)$ is the equilibrium distribution, if and only if the detailed balance equations hold: $\pi_1 = a\pi_2$, $\pi_2 = b\pi_3$, $\pi_3 = c\pi_1$. The first two of the equations yield that $\pi_1 = ab\pi_3$, which can hold together with the third equation only if $abc = 1$. This condition is thus necessary. If $abc = 1$, we find that the detailed balance equations hold for the probability distribution proportional to $(ab, b, 1)$. Thus, the condition $abc = 1$ is also sufficient for reversibility.

2. (5 points=3 points for correct answer + 2 points for proof) Suppose X is a random variable representing the rate of a connection sharing a bufferless link. Suppose the only information known about the probability distribution of X is that $E[X] = 1$ and $P[0 \le X \le 2] = 1$. For a given $s > 0$ fixed, what specific choice of distribution of X meeting these constraints maximizes the equivalent bandwidth $\alpha(s)$, where $\alpha(s) = \frac{\ln(E[e^{sX}])}{s}$?

SOLUTION
Intuitively, X should be as bursty as possible, suggesting the choice $P[X^* = 0] = P[X^* = 2] = 0.5$. Here's a proof. Fix $s > 0$. The function e^{sx} is convex in x over the interval $x \in [0, 2]$, with values 1 and e^{2s} at the endpoints. So $e^{sx} \le 1 + \frac{x}{2}(e^{2s} - 1)$ for $x \in [0, 2]$. Plugging in X on each side, taking expectations, and using the constraint $E[X] = 1$ yields $E[e^{sX}] \le \frac{1+e^{2s}}{2} = E[e^{sX^*}]$.

3. Suppose two queues are served by a constant rate server with service rate C, which serves the longer queue. Specifically, suppose a discrete-time model is used, and after new customers arrive in a given slot, there are C potential services available, which are allocated one at a time to the longer queue. For example, if after the arrivals in a slot there are 9 customers in the first queue, and 7 in the second, and if $C = 4$, then 3 customers are served from the first queue and 1 from the second. Suppose the arrival process A_i to queue i is (σ_i, ρ_i)-upper constrained, for each i, where $\sigma_1, \sigma_2, \rho_1, \rho_2$, and C are strictly positive integers such that $\rho_1 + \rho_2 \le C$.



(a) (3 points) Find a bound on the maximum number of customers in the system, carried over from one slot to the next, which is valid for any order of service within each substream.
(b) (3 points) Find a bound on the maximum delay which is valid for any order of service within each substream. (The bound should be finite if $\rho_1 + \rho_2 < C$.)
(c) (4 points) Suppose in addition that customers within each stream are served in FIFO order. Find an upper bound for the delay of customers in stream 1 which is finite, even if $\rho_1 + \rho_2 = C$. Explain your reasoning.

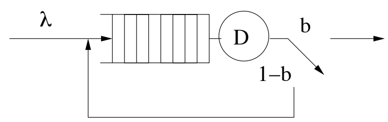
SOLUTION

(a) The two queues combined is equivalent to a single queue system, with a work conserving server that serves up to C customers per slot. The total arrival stream is $(\sigma_1+\sigma_2, \rho_1+\rho_2)$-upper constrained, so that the maximum combined carry over for a given slot t to the next satisfies $q(t) \leq \sigma_1+\sigma_2$.

(b) By the same reason, the delay bound based on system busy periods can be applied to yield $d \leq \lfloor \frac{\sigma_1+\sigma_2-1}{C-\rho_1-\rho_2} \rfloor$.

(c) Under FIFO service within each stream and service to longer queue, each customer from stream 1 exits no later than it would exit from a system with the same arrival streams, FIFO service within each stream, and pure priority to type 2 customers, a case considered in the notes and class. The delay is thus bounded above by $\frac{\sigma_1+\sigma_2}{C-\rho_2}$.

4. Consider a queue with feedback as shown, where $\lambda, D > 0$ and $0 \leq b \leq 1$. New arrivals occur according to a Poisson process of rate λ. The service time of a customer for each visit to the queue is the constant D. Upon a service completion, the customer is returned to the queue with probability $1-b$.



(a)(2 points) Under what condition is the system stable? Justify your answer.

(b)(5 points) Suppose the service order is FIFO, except that a returning customer is able to bypass all other customers and begin a new service immediately. Denote this by PR, for priority to returning, service order. Express the mean total system time of a customer, from the time it arrives until the time it leaves the server for the last time, in terms of λ, b, and D. (Hint: A geometrically distributed random variable with parameter p has first moment $\frac{1}{p}$ and second moment $\frac{2-p}{p^2}$.)

(c)(3 points) If instead the service within the queue were true FIFO, so that returning customers go to the end of the line, would the mean total time in the system be larger, equal, or smaller than for PR service order? Would the variance of the total time in the system be larger, equal, or smaller than for PR service order?

SOLUTION (a) The system is stable if and only if $\frac{\lambda D}{b} < 1$. See part (b) for justification.

(b) Under the PR service order, we can view the network as an ordinary M/GI/1 queue in which the service time variable X for a typical customer is D times a Geo(b) random variable. The system is stable if $\rho < 1$, were $\rho = \lambda\overline{X} = \frac{\lambda D}{b}$. Under this condition, the mean time in the system is given by the Polleczek-Khinchine formula with $\rho = \frac{\lambda D}{b}$, $\overline{X} = \frac{D}{b}$, and $\overline{X^2} = \frac{D^2(2-b)}{b^2}$:

$$T = W + \overline{X} = \frac{\lambda\overline{X^2}}{2(1-\rho)} + \overline{X} = \frac{(2-\lambda D)D}{2(b-\lambda D)}.$$

(c) The distribution of the number of customers in the system does not depend on the order of service (assuming no advance knowledge of how many times a given customer will return). So the mean number of customers in the system does not depend on the service order. So by Little's law, the mean time in the system does not depend on the order of service. However, the FIFO order at the queue would lead to a larger variance of total system time than the PR service order. (This is implied by the fact that if $a_1 < \cdots < a_n$ and $d_1 < \cdots < d_n$ and $a_i < d_i$, then $\sum(d_i - a_i)^2 < \sum(d_{\sigma_i} - a_i)^2$ for any permutation σ other than the identity permutation. Note that the PR order is not FIFO for the queue, but is FIFO for the system.)