University of Illinois at Urbana-Champaign

ECE 467: Communication Network Analysis

Fall 2006
Final Exam

Wednesday, December 13, 2006

Name: ____

- You have three hours for this exam. The exam is closed book and closed note, except that you may consult both sides of three sheets of notes, typed in font size 10 or equivalent handwriting size.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. __________ (15 pts.)

2. __________ (5 pts.)

3. __________ (5 pts.)

4. __________ (10 pts.)

5. __________ (10 pts.)

6. __________ (20 pts.)

7. __________ (15 pts.)

Total: __________(80 pts.)

1. (15 points) Let $M = K \times L$, where K and L are strictly positive integers, and let $\lambda_S, \lambda_F \geq 0$. A bank of M channels (i.e. servers) is shared between slow and fast connections. A slow connection requires the use of one channel and a fast connection requires the use of L channels. Connection requests of each type arrive according to independent Poisson processes, with rates λ_S and λ_F respectively. If there are enough channels available to handle a connection request, the connection is accepted, service begins immediately, and the service time is exponentially distributed with mean one. Otherwise, the request is blocked and cleared from the system.

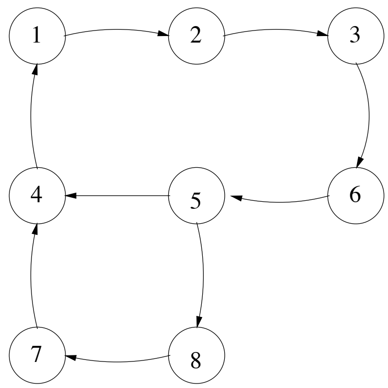
(a) Sketch a transition rate diagram for a continuous-time Markov model of the system.

(b) Explain why there is a simple expression for the equilibrium probability distribution, and find it. (You don't need to find an explicit expression for the normalizing constant.)

(c) Express B_S, the blocking probability for slow requests, and B_F, the blocking probability for fast requests, in terms of the equilibrium distribution.

2. (5 points) Let f and g be two different probability density functions with support $\mathbb{R}_+$, and with the same finite mean, m_1. Let f_L and g_L denote the corresponding sampled lifetime densities: $f_L(x) = \frac{xf(x)}{m_1}$ and $g_L(x) = \frac{xg(x)}{m_1}$. Show that it is impossible that $f_L \prec g_L$. (We write $f_L \prec g_L$ to denote that a random variable with pdf f_L is stochastically smaller than a random variable with pdf g_L.)

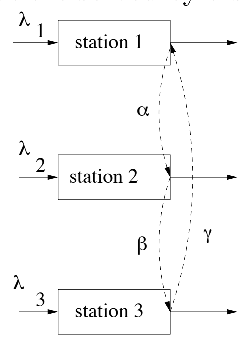
3. (5 points) Consider a discrete-time Markov process with the nonzero one-step transition probabilities indicated by the following graph.



(a) What is the period of state 4?

(b) What is the period of state 6?

4. (10 points) Consider three stations that are served by a single rotating server, as pictured.

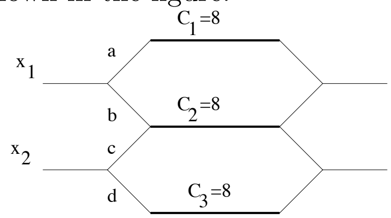


Customers arrive to station i according to a Poisson process of rate λ_i for $1 \leq i \leq 3$, and the total service requirement of each customer is exponentially distributed, with mean one. The rotation of the server is modelled by a three state Markov process with the transition rates α, β, and γ as indicated by the dashed lines. When at a station, the server works at unit rate, or is idle if the station is empty. If the service to a customer is interrupted because the server moves to the next station, the service is resumed when the server returns.

(a) Under what condition is the system stable? Briefly justify your answer.

(b) Identify a method for computing the mean customer waiting time at station one.

5. (10 points) Consider the joint routing and congestion control problem for two users, each of which can split flow over two routes, as shown in the figure.

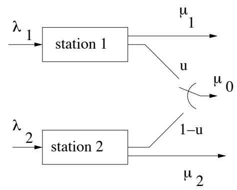


The total flow of user one is $x_1 = a + b$ and the total flow of user 2 is $x_2 = c + d$. The three central links each have capacity 8, and all other links have much larger capacity. User 1 derives value $U_1(a + b) = 2\sqrt{a + b}$ and user 2 derives value $U_2(c + d) = \ln(c + d)$. Let variables p_1, p_2, p_3 denote prices for the three links (i.e. Lagrange multipliers for the capacity constraints).

(a) Write down the optimality conditions for $a, b, c, d, p_1, p_2, p_3$.

(b) Identify the optimal assignment (a, b, c, d) and the corresponding three link prices.

6. (20 points) Consider a system of two service stations as pictured.



Station i has Possion arrivals at rate λ_i and an exponential type server, with rate $m_i(u)$, where $m_1(u) = \mu_1 + u\mu_0$ and $m_2(u) = \mu_2 + (1-u)\mu_0$ and u is a control variable with $u \in U = [0,1]$. Suppose we are interested in the infinite horizon discounted average number of customers in the system, with discount rate $\alpha > 0$.

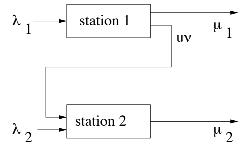
(a) Specify the transition probability matrix, $P(u)$, the interjump CDF, F, and cost function g, for a dynamic programming model.

(b) Write the dynamic programming update equations for the optimal cost-to-go functions, V_n.

(c) Express the optimal state feedback control for n-steps, $u_n^*(x)$, in terms of V_n.

(d) Consider the symmetric case: $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$. Give an educated guess regarding the structure of the optimal control, and give an equivalent condition on V_n. (Note: An equivalent condition, not just a sufficient condition, is asked for here.)

7. (15 points) Let $(\lambda_1, \lambda_2, \nu, \mu_1, \mu_2)$ be a vector of strictly positve parameters, and consider a system of two service stations with transfers as pictured.



Station i has Possion arrivals at rate λ_i and an exponential type server, with rate μ_i. In addition, customers are transferred from station 1 to station 2 at rate $u\nu$, where u is a constant with $u \in U = [0, 1]$. (Rather than applying dynamic programming here, we will apply the method of Foster-Lyapunov stability theory in continuous time.) The system is described by a continuous-time Markov process on $\mathbb{Z}_+^2$ with some transition rate matrix Q. (You don't need to write out Q.)

(a) Under what condition on $(\lambda_1, \lambda_2, \nu, \mu_1, \mu_2)$ is there a choice of the constant u such that the Markov process describing the system is positive recurrent?

(b) Let V be the quadratic Lyapunov function, $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2}$. Compute the drift function QV.

(c) Under the condition of part (a), and using the moment bound associated with the Foster-Lyapunov criteria, find an upper bound on the mean number in the system in equilibrium, $\overline{X_1} + \overline{X_2}$. (The smaller the bound the better.)