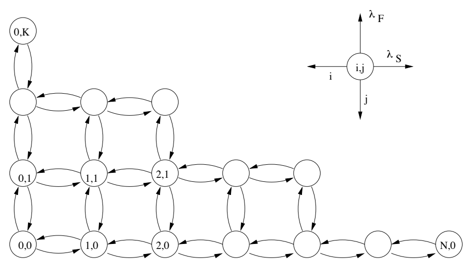
Solutions to final exam. ECE 567 Fall 2006. B. Hajek, Instructor

1. (15 points) Let $M = K \times L$, where K and L are strictly positive integers, and let $\lambda_S, \lambda_F \geq 0$. A bank of M channels (i.e. servers) is shared between slow and fast connections. A slow connection requires the use of one channel and a fast connection requires the use of L channels. Connection requests of each type arrive according to independent Poisson processes, with rates λ_S and λ_F respectively. If there are enough channels available to handle a connection request, the connection is accepted, service begins immediately, and the service time is exponentially distributed with mean one. Otherwise, the request is blocked and cleared from the system.
(a) Sketch a transition rate diagram for a continuous-time Markov model of the system.
(b) Explain why there is a simple expression for the equilibrium probability distribution, and find it. (You don't need to find an explicit expression for the normalizing constant.)
(c) Express B_S, the blocking probability for slow requests, and B_F, the blocking probability for fast requests, in terms of the equilibrium distribution.

SOLUTION
(a)

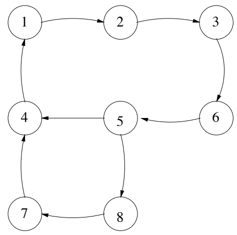


(b) If there were infinitely many channels, the numbers of slow and fast connections would form independent $M/M/\infty$ systems, which is reversible, because each coordinate process is a one-dimensional, and hence reversible, Markov process, and two independent reversible processes are jointly reversible. The equilibrium distribution would be $\pi(i,j) = \frac{e^{-\lambda_S}{\lambda_S}^i}{i!}\frac{e^{-\lambda_F}{\lambda_F}^j}{j!}$. The process with the finite number of channels is obtained by truncation of the process with infinitely many channels, to the state space $\mathcal{S}^{K,L} = \{(i,j) \in \mathbb{Z}_+^2 : i + Lj \leq KL\}$, and is hence also reversible. The equilibrium distribution is $\pi^{K,L}(i,j) = \frac{{\lambda_S}^i{\lambda_F}^j}{i!j!Z(K,L)}$, where $Z(K,L)$ is selected to make $\pi^{K,L}$ sum to one.
(c) $B_S = \sum_{j=0}^{L} \pi^{K,L}((K-j)L, j)$, and $B_F = 1 - \frac{Z(K-1,L)}{Z(K,L)}$.

2. (5 points) Let f and g be two different probability density functions with support $\mathbb{R}_+$, and with the same finite mean, m_1. Let f_L and g_L denote the corresponding sampled lifetime densities: $f_L(x) = \frac{xf(x)}{m_1}$ and $g_L(x) = \frac{xg(x)}{m_1}$. Show that it is impossible that $f_L \prec g_L$. (We write $f_L \prec g_L$ to denote that a random variable with pdf f_L is stochastically smaller than a random variable with pdf g_L. Note that the mean of f_L does not have to equal the mean of g_L.)

SOLUTION
If $f_L \prec g_L$, then $\int_0^\infty \phi(x)f_L(x)dx > \int_0^\infty \phi(x)g_L(x)dx$ for any strictly decreasing positive function ϕ on $\mathbb{R}_+$. Taking $\phi(x) = \frac{1}{x}$ yields that $1 = \int_0^\infty f(x)dx > \int_0^\infty g(x)dx = 1$, which is impossible.

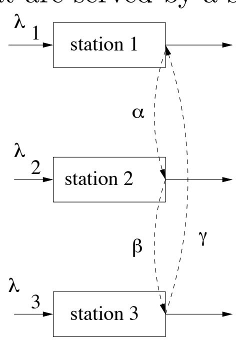
3. (5 points) Consider a discrete-time Markov process with the nonzero one-step transition probabilities indicated by the following graph.



(a) What is the period of state 4?
(b) What is the period of state 6?

SOLUTION
(a) The period of state 4 is $GCD\{4, 6, 8, 10, 12, 14, \ldots\} = GCD\{4, 6\} = 2$.
(b) The process is irreducible, so all states have the same period. So state 6 must also have period 2.

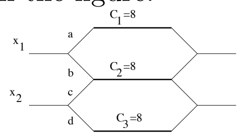
4. (10 points) Consider three stations that are served by a single rotating server, as pictured.



Customers arrive to station i according to a Poisson process of rate λ_i for $1 \leq i \leq 3$, and the total service requirement of each customer is exponentially distributed, with mean one. The rotation of the server is modelled by a three state Markov process with the transition rates α, β, and γ as indicated by the dashed lines. When at a station, the server works at unit rate, or is idle if the station is empty. If the service to a customer is interrupted because the server moves to the next station, the service is resumed when the server returns.
(a) Under what condition is the system stable? Briefly justify your answer.
(b) Identify a method for computing the mean customer waiting time at station one.

SOLUTION
(a) The equilibrium distribution for the server is $\pi = (\pi_1, \pi_2, \pi_3) = (\frac{1}{\alpha Z}, \frac{1}{\beta Z}, \frac{1}{\gamma Z})$, where $Z = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, and the service capacity of station i is thus π_i, for $1 \leq 1 \leq 3$.
(b) The equilibrium distributions for the individual stations can be found one at a time, by the method of phases (even though the stations are not independent). For a fixed station i, the state space would be $\{(l, \theta) : l \geq 0, 1 \leq \theta \leq 3\}$, where l is the number of customers at station i and θ is the phase of the server.

5. (10 points) Consider the joint routing and congestion control problem for two users, each of which can split flow over two routes, as shown in the figure.



The total flow of user one is $x_1 = a + b$ and the total flow of user 2 is $x_2 = c + d$. The three central links each have capacity 8, and all other links have much larger capacity. User 1 derives value $U_1(a + b) = 2\sqrt{a + b}$ and user 2 derives value $U_2(c + d) = \ln(c + d)$. Let variables p_1, p_2, p_3 denote prices for the three links (i.e. Lagrange multipliers for the capacity constraints).

(a) Write down the optimality conditions for $a, b, c, d, p_1, p_2, p_3$.

(b) Identify the optimal assignment (a, b, c, d) and the corresponding three link prices.

SOLUTION

(a) The primal variables should be feasible: $a, b, c, d \geq 0$ and $a \leq C_1, b + c \leq C_2, d \leq C_3$. The dual variables should satisfy the positivity and complementary slackness condition: $p_i \geq 0$ with equality if link i is not saturated. It is easy to see that the optimal flow will saturate all three links, so that nonzero prices are permitted. Finally, since $U_1'(x_1) = x_1^{-1/2}$ and $U_2'(x_2) = \ln(x_2)$, the remaining conditions are:

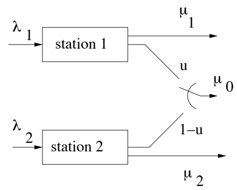
$$\begin{aligned} (a+b)^{-1/2} &\leq p_1 \quad \text{with equality if } a > 0 \\ (a+b)^{-1/2} &\leq p_2 \quad \text{with equality if } b > 0 \\ \tfrac{1}{c+d} &\leq p_2 \quad \text{with equality if } c > 0 \\ \tfrac{1}{c+d} &\leq p_3 \quad \text{with equality if } d > 0 \end{aligned}$$

(b) Clearly we can set $a = d = 8$ and $c = 8 - b$. It remains to find b. The optimality conditions become

$$\begin{aligned} (8+b)^{-1/2} &\leq p_2 \quad \text{with equality if } b > 0 \\ \tfrac{1}{16-b} &\leq p_2 \quad \text{with equality if } b < 8 \end{aligned}$$

Since $(8+b)^{-1/2} > \frac{1}{16-b}$ over the entire range $0 \leq b \leq 8$, the optimal choice of b is $b = 8$. This yields the assignment $(a, b, c, d) = (8, 8, 0, 8)$, price vector $(p_1, p_2, p_3) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{8})$, and maximum value $2 + \ln(8)$.

6. (20 points) Consider a system of two service stations as pictured.



Station i has Possion arrivals at rate λ_i and an exponential type server, with rate $m_i(u)$, where $m_1(u) = \mu_1 + u\mu_0$ and $m_2(u) = \mu_2 + (1-u)\mu_0$ and u is a control variable with $u \in U = [0,1]$. Suppose we are interested in the infinite horizon discounted average number of customers in the system, with discount rate $\alpha > 0$.
(a) Specify the transition probability matrix, $P(u)$, the interjump CDF, F, and cost function g, for a dynamic programming model.
(b) Write the dynamic programming update equations for the optimal cost-to-go functions, V_n.
(c) Express the optimal state feedback control for n-steps, $u_n^*(x)$, in terms of V_n.
(d) Consider the symmetric case: $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$. Give an educated guess regarding the structure of the optimal control, and give an equivalent condition on V_n. (Note: An equivalent condition, not just a sufficient condition, is asked for here.)

SOLUTION
(a) We use $\mathcal{S} = \mathbb{Z}_+^2$, where state $x = (x_1, x_2)$ denotes x_i customers in station i, for $i = 1, 2$. Let $A_1(x_1, x_2) = (x_1 + 1, x_2)$ and $D_1(x_1, x_2) = ((x_1 - 1)_+, x_2)$, and define A_2 and D_2 similarly. Then

$$p_{x,y}(u) = \frac{1}{\gamma} \sum_{i=1}^{2} \left(\lambda_i I_{\{y=A_i(x)\}} + m_i(u) I_{\{y=D_i(x)\}} \right)$$

The CDF F corresponds to the exponential distribution with parameter $\gamma = \lambda_1 + \lambda_2 + \mu_0 + \mu_1 + \mu_2$, and the instantaneous cost function is $g(x) = x_1 + x_2$.
(b) The backwards equation of dynamic programming becomes

$$V_{n+1}(x) = g(x) + \min_{0 \le u \le 1} \frac{\beta}{\gamma} \sum_{i=1}^{2} (\lambda_i V_n(A_i(x)) + m_i(u) V_n(D_i(x)))$$

or, after plugging in the optimal value for u,

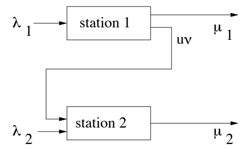
$$V_{n+1}(x) = g(x) + \frac{\beta}{\gamma} \left[\sum_{i=1}^{2} (\lambda_i V_n(A_i(x)) + \mu_i(u) V_n(D_i(x))) + \mu_0 \min\{V_n(D_1(x)), V_n(D_2(x))\} \right]$$

with the intial condition $V_0 \equiv 0$.
(c)

$$u_n^*(x) = \begin{cases} 1 & \text{if } V_n(D_1 x) \le V_2(D_2 x) \\ 0 & \text{else.} \end{cases} \tag{1}$$

(d) We conjecture that the service rate μ_0 should be allocated to the station with the longer queue. That is, $u_n^*(x) = I_{\{x_1 \ge x_2\}}$. Equivalently, $V(x_1, x_2) \le V(x_1 - 1, x_2 + 1)$ whenever $0 \le x_1 \le x_2$ and $V(x_1, x_2) \le V(x_1 + 1, x_2 - 1)$ whenever $0 \le x_2 \le x_1$.

7. (15 points) Let $(\lambda_1, \lambda_2, \nu, \mu_1, \mu_2)$ be a vector of strictly positve parameters, and consider a system of two service stations with transfers as pictured.



Station i has Possion arrivals at rate λ_i and an exponential type server, with rate μ_i. In addition, customers are transferred from station 1 to station 2 at rate $u\nu$, where u is a constant with $u \in U = [0, 1]$. (Rather than applying dynamic programming here, we will apply the method of Foster-Lyapunov stability theory in continuous time.) The system is described by a continuous-time Markov process on $\mathbb{Z}_+^2$ with some transition rate matrix Q. (You don't need to write out Q.)
(a) Under what condition on $(\lambda_1, \lambda_2, \nu, \mu_1, \mu_2)$ is there a choice of the constant u such that the Markov process describing the system is positive recurrent?
(b) Let V be the quadratic Lyapunov function, $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2}$. Compute the drift function QV.
(c) Under the condition of part (a), and using the moment bound associated with the Foster-Lyapunov criteria, find an upper bound on the mean number in the system in equilibrium, $\overline{X_1} + \overline{X_2}$. (The smaller the bound the better.)

SOLUTION
(a) System is positive recurrent for some u if and only if $\lambda_1 < \mu_1 + \nu$, $\lambda_2 < \mu_2$, and $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$.

(b)

$$\begin{aligned} QV(x) &= \sum_{y: y \neq x} q_{xy}\,(V(y) - V(x)) \\ &= \frac{\lambda_1}{2}[(x_1+1)^2 - x_1^2] + \frac{\lambda_2}{2}[(x_2+1)^2 - x_2^2] + \frac{\mu_1}{2}[(x_1-1)_+^2 - x_1^2] + \\ &\quad \frac{\mu_2}{2}[(x_2-1)_+^2 - x_2^2] + \frac{u\nu I_{\{x_1 \geq 1\}}}{2}[(x_1-1)^2 - x_1^2 + (x_2+1)^2 - x_2^2] \end{aligned} \tag{2}$$

(c) If the righthand side of (2) is changed by dropping the positive part symbols and dropping the factor $I_{\{x_1 \geq 1\}}$, then it is not increased, so that

$$\begin{aligned} QV(x) &\leq x_1(\lambda_1 - \mu_1 - u\nu) + x_2(\lambda_2 + u\nu - \mu_2) + K \\ &\leq -(x_1 + x_2)\min\{\mu_1 + u\nu - \lambda_1, \mu_2 - \lambda_2 - u\nu\} + K \end{aligned} \tag{3}$$

where $K = \frac{\lambda_1+\lambda_2+\mu_1+\mu_2+2\nu}{2}$. To get the best bound on $\overline{X_1} + \overline{X_2}$, we select u to maximize the min term in (3), or $u = u^*$, where u^* is the point in $[0, 1]$ nearest to $\frac{\mu_1+\mu_2-\lambda_1-\lambda_2}{2\nu}$. For $u = u^*$, we find $QV(x) \leq -\epsilon(x_1 + x_2) + K$ where $\epsilon = \min\{\mu_1 + \nu - \lambda_1, \mu_2 - \lambda_2, \frac{\mu_1+\mu_2-\lambda_1-\lambda_2}{2}\}$. (Which of the three terms is smallest in the expression for ϵ corresponds to the three cases $u^* = 1$, $u^* = 0$, and $0 < u^* < 1$, respectively.)

Remark: It is easy to check that this same ϵ is the largest constant such that the stability conditions (with strict inequality relaxed to less than or equal) hold with (λ_1, λ_2) replaced by $(\lambda_1 + \epsilon, \lambda_2 + \epsilon)$.