ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
PROBLEM SET 1 **Wednesday, September 6**

Topic: Discrete-state Markov processes including Poisson processes

Assigned reading: Sections 1.1-1.5 of the notes. (Kleinrock, Queueing Systems Vol. I , Sections 2-2-2.5, has related material.)

Problems to be handed in:

1. Poisson merger
Summing counting processes corresponds to "merging" point processes. Show that the sum of K independent Poisson processes, having rates $\lambda_1, \ldots, \lambda_K$, respectively, is a Poisson process with rate $\lambda_1 + \ldots + \lambda_K$. (Hint: First formulate and prove a similar result for sums of random variables, and then think about what else is needed to get the result for Poisson processes. You can use any one of the equivalent definitions given by Proposition 1.5.1 in the notes. Don't forget to check required independence properties.)

2. Poisson splitting
Consider a stream of customers modeled by a Poisson process, and suppose each customer is one of K types. Let $(p_1, \ldots, p_K)$ be a probability vector, and suppose that for each k, the k^{th} customer is type i with probability p_i. The types of the customers are mutually independent and also independent of the arrival times of the customers. Show that the stream of customers of a given type i is again a Poisson stream, and that its rate is λp_i. (Same hint as in the previous problem applies.) Show furthermore that the K substreams are mutually independent.

3. Poisson method for coupon collector's problem
(a) Suppose a stream of coupons arrives according to a Poisson process $(A(t) : t \geq 0)$ with rate $\lambda = 1$, and suppose there are k types of coupons. (In network applications, the coupons could be pieces of a file to be distributed by some sort ot gossip algorithm.) The type of each coupon in the stream is randomly drawn from the k types, each possibility having probability $\frac{1}{k}$, and the types of different coupons are mutually independent. Let $p(k, t)$ be the probability that at least one coupon of each type arrives by time t. (The letter "p" is used here because the number of coupons arriving by time t has the Poisson distribution). Express $p(k, t)$ in terms of k and t.
(b) Find $\lim_{k\to\infty} p(k, k \ln k + kc)$ for an arbitrary constant c. That is, find limit of the probability that the collection is complete at time $t = k \ln k + kc$. (Hint: If $a_k \to a$ as $k \to \infty$, then $(1 + \frac{a_k}{k})^k \to e^a$.)
(c) The rest of this problem shows that the limit found in part (b) also holds if the total number of coupons is deterministic, rather than Poisson distributed. One idea is that if t is large, then $A(t)$ is not too far from its mean with high probability. Show, specifically, that
$\lim_{k\to\infty} P[A(k \ln k + kc) \geq k \ln k + kc'] = \begin{cases} 0 & \text{if } c < c' \\ 1 & \text{if } c > c' \end{cases}$
(d) Let $d(k, n)$ denote the probability that the collection is complete after n coupon arrivals. (The letter "d" is used here because the number of coupons, n, is deterministic.) Show that for any k, t, and n fixed, $d(k, n)P[A(t) \geq n] \leq p(k, t) \leq P[A(t) \geq n] + P[A(t) \leq n]d(k, n)$.
(e) Combine parts (c) and (d) to identify $\lim_{k\to\infty} d(k, k \ln k + kc)$.

4. The sum of a random number of random variables

Let $(X_i : i \geq 0)$ be a sequence of independent and identically distributed random variables. Let $S = X_1 + \ldots + X_N$, where N is a nonnegative random variable that is independent of the sequence. (a) Express $E[S]$ and $Var(S)$ in terms of the mean and variance of X_1 and N. (Hint: Use the fact $E[S^k] = E[E[S^k|N]]$). (b) Express the characteristic function Φ_S in terms of the characteristic function Φ_X of X_1 and the z-transform $B(z)$ of N.

5. Mean hitting time for a simple Markov process

Let $(X(n) : n \geq 0)$ denote a discrete-time, time-homogeneous Markov chain with state space $\{0, 1, 2, 3\}$ and one-step transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-a & 0 & a & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for some constant a with $0 \leq a \leq 1$. (a) Sketch the transition probability diagram for X and give the equilibrium probability vector. If the equilibrium vector is not unique, describe all the equilibrium probability vectors.

(b) Compute $E[\min\{n \geq 1 : X(n) = 3\}|X(0) = 0]$.

6. A two station pipeline in continuous time

Consider a pipeline consisting of two single-buffer stages in series. Model the system as a continuous-time Markov process. Suppose new packets are offered to the first stage according to a rate λ Poisson process. A new packet is accepted at stage one if the buffer in stage one is empty at the time of arrival. Otherwise the new packet is lost. If at a fixed time t there is a packet in stage one and no packet in stage two, then the packet is transfered during $[t, t+h)$ to stage two with probability $h\mu_1 + o(h)$. Similarly, if at time t the second stage has a packet, then the packet leaves the system during $[t, t+h)$ with probability $h\mu_2 + o(h)$, independently of the state of stage one. Finally, the probability of two or more arrival, transfer, or departure events during $[t, t+h)$ is $o(h)$. (a) What is an appropriate state-space for this model? (b) Sketch a transition rate diagram. (c) Write down the Q matrix. (d) Derive the throughput, assuming that $\lambda = \mu_1 = \mu_2 = 1$. (e) Still assuming $\lambda = \mu_1 = \mu_2 = 1$. Suppose the system starts with one packet in each stage. What is the expected time until both buffers are empty?

7. Simple population growth models

A certain population of living cells begins with a single cell. Each cell present at a time t splits into two cells during the interval $[t, t+h]$ with probability $\lambda h + o(h)$, independently of the other cells, for some constant $\lambda > 0$. The number of cells at time t, $N(t)$, can thus be modeled as a continuous-time Markov chain. (a) Sketch an appropriate transition rate diagram and describe the transition rate matrix. (b) For fixed $t \geq 0$, let $P(z, t)$ denote the z-transform of the probability vector $\pi(t)$. Starting with the Kolmogorov forward equations for $(\pi(t))$, derive a partial differential equation for the z-transform of $\pi(t)$. Show that the solution is

$$P(z, t) = \frac{ze^{-\lambda t}}{1 - z + ze^{-\lambda t}}$$

(c) Find E[N(t)]. (d) Solve for $\pi(t)$. (e) Another population evolves deterministically. The first cell splits at time λ^{-1}, and thereafter each cell splits into two exactly λ^{-1} time units after its creation. Find the size of the population of this population as a function of time, and compare to your answer in part (c).

8. Equilibrium distribution of the jump chain

Suppose that π is the equilibrium distribution for a time-homogeneous Markov process with transition rate matrix Q. Suppose that $B^{-1} = \sum_i -q_{ii}\pi_i$, where the sum is over all i in the state space, is finite. Show that the equilibrium distribution for the jump chain $(X^J(k) : k \geq 0)$ is given by $\pi_i^J = -Bq_{ii}\pi_i$. (So π and π^J are identical if and only if q_{ii} is the same for all i.)