ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
PROBLEM SET 2 **Due Wednesday, September 20**

Topics: renewal theory, classification of Markov processes, Markov queueing systems, method of phases

Assigned reading: Sections 1.7-1.13 of the notes. Additional reading can be found in Kleinrock, Vol. 1, Section 2.1, pp. 10-19; Chapter 3 pp. 89-110; Sect. 4.1 pp. 115-119; and Section 5.2 169-174. Bertsekas and Gallager, Chap. 3 through Sect. 3.4.3

1. A simple Poisson process calculation

1. Let $(N(t) : t \geq 0)$ be a Poisson random process with rate $\lambda > 0$. Compute $P[N(s) = i|N(t) = k]$ where $0 < s < t$ and i and k are nonnegative integers. (Caution: note order of s and t carefully).

2. An alternating renewal process

Consider a two-color traffic signal on a street corner which alternately stays red for one minute and then stays green for one minute. Suppose you are riding your bicycle towards the signal and are a block away when you first spot it, so that it will take you one-half minute to reach the signal. (a) What is the probability that the signal will change colors (at least once) before you reach the corner? Explain. (b) Compute the conditional expectation of how long you must wait at the corner (possibly not at all) given that the light is green when you first spot it. Repeat assuming the light is red when you first spot it. (Which is greater? What is the average of the two answers?) (c) Repeat part (a) if instead the signal randomly changes colors, staying the same color for a time period uniformly distributed between 0 and 2 minutes. The times between switches are assumed to be independent. (d) How fast are you riding? (Hint: What units of speed can you use so that you don't need more information?)

3. A mean hitting time problem

Let $(X(t) : t \geq 0)$ be a time-homogeneous, pure-jump Markov process with state space $\{0, 1, 2\}$ and Q matrix

$$Q = \begin{pmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 2 & 0 & -2. \end{pmatrix}$$

(a) Write down the state transition diagram and compute the equilibrium distribution.
(b) Compute $a_i = E[\min\{t \geq 0 : X(t) = 1\}|X(0) = i]$ for $i = 0, 1, 2$. If possible, use an approach that can be applied to larger state spaces.
(c) Derive a variation of the Kolmogorov forward differential equations for the quantities: $\alpha_i(t) = P[X(s) \neq 2 \text{ for } 0 \leq s \leq t \text{ and } X(t) = i|X(0) = 0]$ for $0 \leq i \leq 2$. (You need not solve the equations.)
(d) The forward Kolmogorov equations describe the evolution of an initial probability distribution going forward in time, given an initial. In other problems, a boundary condition is given at a final time, and a differential equation working backwards in time from a final condition is called for (called Kolmogorov backward equations). Derive a backward differential equation for: $\beta_j(t) = P[X(s) \neq 2 \text{ for } t \leq s \leq t_f|X(t) = j]$, for $0 \leq j \leq 2$ and $t \leq t_f$ for some fixed time t_f. (Hint: Express $\beta_i(t-h)$ in terms of the $\beta_j(t)'s$ for $t \leq t_f$, and let $h \to 0$. You need not solve the equations.)

4. An unusual birth-death process

Consider the birth-death process X with arrival rates $\lambda_k = (p/(1-p))^k/a_k$ and death rates $\mu_k = (p/(1-p))^{k-1}/a_k$, where $.5 < p < 1$, and $a = (a_0, a_1, \ldots)$ is a probability distribution on the nonnegative integers with $a_k > 0$ for all k. (a) Classify the states for the process X as transient, null-recurrent or positive recurrent. (b) Check that $aQ = 0$. Is a an equilibrium distribution for X? Explain. (c) Find the one-step transition probabilities for the jump-chain, X^J (d) Classify the states for the process X^J as transient, null-recurrent or positive recurrent.

5. A queue with decreasing service rate

Consider a queueing system in which the arrival process is a Poisson process with rate λ. Suppose the instantaneous completion rate is μ when there are K or fewer customers in the system, and $\mu/2$ when there are $K+1$ or more customers in the system. The number in the system is modeled as a birth-death Markov process. (a) Sketch the transition rate diagram. (b) Under what condition on λ and μ are all states positive recurrent? Under this condition, give the equilibrium distribution. (c) Suppose that $\lambda = (2/3)\mu$. Describe in words the typical behavior of the system, given that it is initially empty.

6. Limit of a distrete time queueing system

Model a queue by a discrete-time Markov chain by recording the queue state after intervals of q seconds each. Assume the queue evolves during one of the atomic intervals as follows: There is an arrival during the interval with probability αq, and no arrival otherwise. If there is a customer in the queue at the beginning of the interval then a single departure will occur during the interval with probability βq. Otherwise no departure occurs. Suppose that it is impossible to have an arrival and a departure in a single atomic interval. (a) Find a_k=P[an interarrival time is kq] and b_k=P[a service time is kq]. (b) Find the equilibrium distribution, $p = (p_k : k \geq 0)$, of the number of customers in the system at the end of an atomic interval. What happens as $q \to 0$?

7. An M/M/1 queue with impatient customers

Consider an M/M/1 queue with parameters λ and μ with the following modification. Each customer in the *queue* will defect (i.e. depart without service) with probability $\alpha h + o(h)$ in an interval of length h, independently of the other customers in the queue. Once a customer makes it to the server it no longer has a chance to defect and simply waits until its service is completed and then departs from the system. Let $N(t)$ denote the number of customers in the system (queue plus server) at time t. (a) Give the transition rate diagram and generator matrix Q for the Markov chain $N = (N(t) : t \geq 0)$. (b) Under what conditions are all states positive recurrent? Under this condition, find the equilibrium distribution for N. (You need not explicitly sum the series.) (c) Suppose that $\alpha = \mu$. Find an explicit expression for p_D, the probability that a typical arriving customer defects instead of being served. Does your answer make sense as λ/μ converges to zero or to infinity?

8. Statistical multiplexing

Consider the following scenario regarding a one-way link in a store-and-forward packet communication network. Suppose that the link supports eight connections, each generating traffic at 5 kilobits per second (kbps). The data for each connection is assumed to be in packets exponentially distributed in length with mean packet size 1 kilobit. The packet lengths are assumed mutually independent and the packets for each stream arrive according to a Poisson process. Packets are queued at the beginning of the link if necessary, and queue space is unlimited. Compute the mean delay (queueing plus transmission time–neglect propagation delay) for each of the following three scenarios. Compare your answers. (a) (Full multiplexing) The link transmit speed is 50 kbps. (b) The link is replaced by two 25 kbps links, and each of the two links carries four sessions. (Of course the delay would be larger if the sessions were not evenly divided.) (c) (Multiplexing over two links) The link is replaced by two 25 kbps links. Each packet is transmitted on one link or the other, and neither link is idle whenever a packet from any session is waiting.

9. A queue with blocking

(M/M/1/5 system) Consider an M/M/1 queue with service rate μ, arrival rate λ, and the modification that at any time, at most five customers can be in the system (including the one in service, if any). If a customer arrives and the system is full (i.e. already has five customers in it) then the customer is dropped, and is said to be blocked. Let $N(t)$ denote the number of customers in the system at time t. Then $(N(t) : t \geq 0)$ is a Markov chain. (a) Indicate the transition rate diagram of the chain and find the equilibrium probability distribution. (b) What is the probability, p_B, that a typical customer is blocked? (c) What is the mean waiting time in queue, W, of a typical customer that is not blocked? (d) Give a simple method to numerically calculate, or give a simple expression for, the mean length of a busy period of the system. (A busy period begins with the arrival of a customer to an empty system and ends when the system is again empty.)

10. Multiplexing circuit and packet data streams

Consider a communication link that is shared between circuit switched traffic (consisting of "calls") and datagram packet traffic. Suppose new calls arrive at rate λ_c, that new packets arrive at rate λ_p, and that the call durations are exponential with parameter δ_c. All calls are "served" simultaneously. Suppose that the link capacity is exactly sufficient to carry C calls, and that calls have priority over packets. Thus, a new call is blocked and lost if and only if it arrives to find C calls already in progress. Packets are not blocked, but instead they are queued until transmission. Finally, suppose that the instantaneous service rate of packets is $\mu(C - n_c)$ when there are n_c calls in progress. (a) Define a continuous-time, countable state Markov chain to model the system, and indicate the transition rates. (b) Is the method of Markov processes with phases relevant to this problem? If so, describe in detail how the method applies. (c) What is the necessary and sufficient condition for the system to be stable (be as explicit as possible).

11. On two distibutions seen by customers

Consider a queueing system in which the number in the system only changes in steps of plus one or minus one. Let $D(k,t)$ denote the number of customers that depart in the interval [0,t] that leave behind exactly k customers, and let R(k,t) denote the number of customers that arrive in the interval [0,t] to find exactly k customers already in the system. (a) Show that $|D(k,t) - R(k,t)| \leq 1$ for all k and t. (b) Let α_t (respectively δ_t) denote the number of arrivals (departures) up to time t. Suppose that $\alpha_t \to \infty$ and $\alpha_t/\delta_t \to 1$ as $t \to \infty$. Show that if the following two limits exist for a given value k, then they are equal: $r_k = \lim_{t\to\infty} R(k,t)/\alpha_t$ and $d_k = \lim_{t\to\infty} D(k,t)/\delta_t$.