ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**

PROBLEM SET 4 **Due Wednesday, October 18**

Topics: M/G/1 (with vacation or priority, TDMA/FDMA), G/M/1, G/G/1 queues.

Recommended reading: Kleinrock, Vol. 1, Section 5.3-5.8 pp. 174-216; Chapter 6 pp. 241-259; Sect.8.1 pp. 275-283. Bertsekas and Gallager (Either edition), Section 3.5.

1. A queue with customers arriving in pairs

Customers arrive at a single-server queue two at a time. The arrival instants for pairs of customers are given by a Poisson process with rate λ. The service times of customers are independent, exponentially distributed with parameter μ. Let $N(t)$ denote the number of customers in the system at time t. (a) The process $(N(t) : t \geq 0)$ is a continuous time pure jump Markov process with statespace $\mathbb{Z}_+$. Sketch the transition rate diagram of N. (b) See what can be deduced about N by applying the Foster-Lyapunov stability criteria and related moment bounds for the Lyapunov function $V(x) = x$. (c) Repeat part (b) using $V(x) = \frac{x^2}{2}$.
(d) Solve for the z transform of the equilibrium distribution and find the equilibrium distribution. Under what condition does it exist (i.e. under what condition is N positive recurrent?)
(e) Let $(r_k : k \geq 0)$ and $(d_k : k \geq 0)$ be the equilibrium probabilities defined by

$$\begin{aligned} r_k &= P[\text{there are } k \text{ in the system just before the arrival of a pair of customers}] \\ d_k &= P[\text{there are } k \text{ in the system just after the departure of a typical customer}]. \end{aligned}$$

Express $(r_k : k \geq 0)$ and $(d_k : k \geq 0)$ in terms of $(p_k : k \geq 0)$, λ and μ. In particular, does $r_k = p_k$? or $d_k = p_k$? or $r_k = d_k$?

2. Token bucket regulation of a Poisson stream

A token bucket traffic regulation scheme works as follows. The packet stream to be regulated is modeled as a Poisson stream with arrival rate λ. There is a token pool that holds up to B tokens. When a packet arrives, if there is a token in the token pool then the packet instantly passes through the regulator, and one of the tokens is removed from the token pool. If instead the token pool is empty, then the packet is lost. (Notice that packets are never queued.) New tokens are generated periodically, with one time unit between successive generation times. If a token is generated when the token pool is full, the token is lost. (The token pool acts like a "leaky bucket".) (a) Identify an embedded discrete-time, discrete-state Markov process, and describe the one-step transition probabilities of the chain. (b) Express the fraction of packets lost (long term average) in terms of λ, B and π, where π denotes the equilibrium probability vector for your Markov chain. (You do NOT need to find π.) (c) As an approximation, suppose that the times between new token generations are independent, exponentially distributed with common mean 1. Find a fairly simple expression for the loss probability.

3. Extremality of constant interarrival times for G/M/1 queues

Consider two G/M/1 queueing systems with common service rate μ. The first has a general interarrival distribution function $A(t)$ with mean $1/\lambda$ such that $\lambda/\mu < 1$. The second has constant interarrival times, all equal to $1/\lambda$. (Thus the interarrival distribution function in the second system is $A_d(t) = 0$ if $t < 1/\lambda$ and $A_d(t) = 1$ otherwise, and the system is called a D/M/1 system). (a) Show that the Laplace transforms are ordered: $A_d^*(s) \leq A^*(s)$ for all $s \geq 0$. (b) Show that the mean number in the system at arrival times (in equilibrium) and the mean waiting time in the system is smaller for the D/M/1 system.

4. Propagation of perturbations

Consider a single-server queueing system which is initially empty and for which a total of five customers arrive, at times 1,2,5,13, and 14, respectively. Suppose the amounts of time required to serve the customers are 5,2,4,2,2 time units, respectively. (a) Sketch the unfinished work in the system as a function of time. Indicate the departure times of the five customers, and compute the waiting times in queue of the five customers. (b) Repeat part (a), assuming that the service time of the second customer is increased to 3 time units. (c) Repeat part (a), assuming the service time of the second customer is increased to 4 time units. (d) Describe briefly in words how in general an increase in service time of one customer effects the waiting times of other customers in the system.

5. On priority M/GI/1 queues

Consider an M/GI/1 system with *preemptive resume* priority service. There are assumed to be two priority classes, with independent arrival streams and service requirements. Class 1 has priority over class 2. Customers of the ith class arrive according to a Poisson process of rate λ_i, and the service times of customers of class i are independent and identically distributed. Use X_i to denote a random variable with the same distribution as the service requirements for class i, for $i = 1$ or $i = 2$. (a) Describe what "preemptive resume priority" means. (b) *Derive* the mean time in the *system*, T_1 and T_2, for class 1 and class 2, respectively.

6. Optimality of the μc rule

Consider an $M/G/1$ queue with customer classes 1 through K and a nonpreemptive priority service discipline. Let $\mu_i = 1/\overline{X_i}$ where $\overline{X}_i$ denotes the mean service time for type i customers and let c_i, with $c_i > 0$, denote the cost per unit time of holding a class i customer in the queue. For some permutation $\sigma = (\sigma_1, \sigma_1, \ldots, \sigma_K)$ of $(1, 2, \ldots, K)$, the class σ_1 customers are given the highest priority, the class σ_2 customers are given the next highest priority, and so on. The resulting long run average cost per unit time is $\sum_i \lambda_i c_i W_i(\sigma)$, where $W_i(\sigma)$ is the mean waiting time of class i customers under the priority order σ. (a) Show that an ordering σ minimizes the long-run average cost per unit time over all possible orderings if and only if $\mu_{\sigma_1} c_{\sigma_1} \geq \mu_{\sigma_2} c_{\sigma_2} \geq \ldots \geq \mu_{\sigma_K} c_{\sigma_K}$. (Hint: If σ does not satisfy the given condition, then for some i, $\mu_{\sigma_i} c_{\sigma_i} < \mu_{\sigma_{i+1}} c_{\sigma_{i+1}}$. Appeal to the conservation of work equation to argue that by swapping σ_i and σ_{i+1}, an ordering $\hat{\sigma}$ with strictly smaller cost than σ is obtained. You still also need to figure out how to prove both the "if" and "only if" portions of the statement.) (b) Write a brief intuitive explanation for the result of part (a).

7. A discrete time M/GI/1 queue

Consider a single server queue in discrete time. Suppose that during each time slot, one customer arrives with probability p, and no customers arrive with probability $1 - p$. The arrival events for different slots are mutually independent. The service times are independent of the arrival times and of each other, and the service time of any given customer is equally likely to be 1,2,3,4,5 or 6 slots. Find W, the mean number of complete slots spent in the queue by a typical customer (don't include time in service). Show your reasoning. Also identify an embedded Markov chain for the system.