ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
PROBLEM SET 6 **Due Wednesday, November 15**

Topics: Graph Algorithms, Routing and Flow Control

Assigned reading: Bertsekas and Gallager, either edition, Sections 5.1 (skim), 5.2, 5.4, 5.5, and 5.6, and Chapter 6 (skim), in addition to class notes.

1. A shortest path problem
A weighted graph with a single source node s is shown in Fig. 1. The weight of each edge is assumed to be the same in either direction. (a) Indicate the tree of shortest paths from s to all other nodes found by Dijkstra's shortest path algorithm. (b) Is the tree of shortest paths from s for the graph unique? *Justify your answer.* (c) How many iterations of the synchronous Bellman-Ford algorithm are required to find shortest length paths from the source node s to all other nodes of this graph?

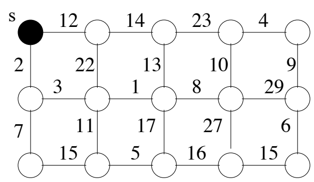

Figure 1: An undirected weighted graph.

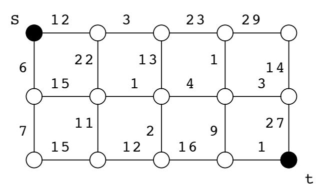

Figure 2: A weighted graph with terminals s and t.

2. A minumum weight spanning tree problem
Consider again the graph of Figure 1. Find the minimum weight (undirected) spanning tree found by Dijkstra's MWST algorithm. Is the MWST for this graph unique? *Justify your answer.*

3. A maximum flow problem
A flow graph with a source node s and a terminal node t is indicated in Figure 2. A line between a pair of nodes indicates a pair of directed links, one in each direction. The capacity of each directed link in a pair is the same, and is indicated. Find a maximum value $s - t$ flow and prove that it is indeed a maximum flow.

4. A flow deviation problem (Frank-Wolfe method)
Consider the communication network with 24 links indicated in Figure 3. Each undirected link in the figure represents two directed links, one in each direction. There are four origin destination pairs: $W = \{(1,9),(3,7),(9,1),(7,3)\}$, each with demand b. The initial routing is deterministic, being concentrated on the four paths $(1,2,5,6,9)$,$(3,6,5,8,7)$,$(9,8,5,4,1)$, and $(7,4,5,2,3)$. Suppose that the cost associated with any one of the 24 links is given by $D(F) = F^2/2$, where F is the total flow on the link, measured in units of traffic per second.
(a) What is the cost associated with the initial routing?
(b) Describe the flow (for all four destinations) after one iteration of the flow deviation algorithm. Assume that, for any given origin-destination pair, any route from the origin to the destination can be used. Is the

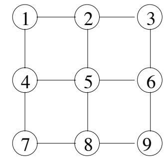

Figure 3: Mesh network with one source and one destination

resulting flow optimal?

5. A simple routing problem in a queueing network
Consider the open queueing network with two customer classes shown in Figure 4. Customers of each type

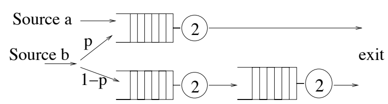

Figure 4: Open network with two types of customers

arrive according to a Poisson process with rate one. All three stations have a single exponential server with rate two, all service times are independent, and the service order is first-come, first-served. Each arrival of a customer of type b is routed to the upper branch with probability p, independently of the history of the system up to the time of arrival. (a) Find expressions for $D_a(p)$ and $D_b(p)$, the mean time in the network for type a and type b customers, respectively. (b) Sketch the set of possible operating points $\{(D_a(p), D_b(p)) : 0 \leq p \leq 1\}$. (c) Find the value p_a of p which minimizes $D_a(p)$. (d) Find the value p_b of p which minimizes $D_b(p)$. (e) Find the value p_{ave} of p which minimizes the average delay, $(D_a(p) + D_b(p))/2$. Also, how many iterations are required for the flow deviation algorithm (Frank-Wolfe method) to find p_{ave}? (f) Find the value p_m of p which minimizes $\max\{D_a(p), D_b(p)\}$.

6. A joint routing and congestion control problem
Consider the network shown in Figure 5 and suppose (1,4) and (2,3) are the only two origin-destination pairs

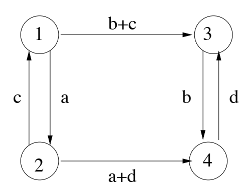

Figure 5: Network with joint routing and congestion control

to be supported by the network. Suppose that the system cost for flow F on any link l is $D_l(F) = \frac{F^2}{2}$, and that the utility of flow r_{ij} for origin-destination pair (i,j) is $\beta_{ij}\log(r_{ij})$ for constants $\beta_{14}, \beta_{23} > 0$. Suppose that for each of the two origin-destination pairs that flow can be split between two paths. For convenience of notation we use the letters a, b, c, d to denote the values of the four path flows: $a = x_{124}$, $b = x_{134}$, $c = x_{213}$, and $d = x_{243}$. The joint routing and congestion control problem is to minimize the sum of link costs minus the sum of the utilities:

$$\min_{a,b,c,d\geq 0} \frac{1}{2}(a^2 + b^2 + c^2 + d^2 + (b+c)^2 + (a+c)^2) - \beta_{14}\log(a+b) - \beta_{23}\log(c+d)$$

(a) Write out the optimality conditions. Include the possibility that some flow values may be zero. (b) Show that if all flow values are positive, then $a = b$ and $c = d$. (c) Find the optimal flows if $\beta_{14} = 66$ and $\beta_{23} = 130$. Verify that for each route used, the price of the route (sum of $D'(F_l)$ along the route) is equal to the marginal utility of flow for the origin-destination of the route.

7. Sufficiency of the optimality condition for hard link constraints
Consider a network with link set L, users indexed by a set of fixed routes R, and hard capacity constraints. Use the following notation:

- $A = (A_{lr} : l \in L, r \in R)$ with $A_{lr} = 1$ if link l is in route r, and $A_{lr} = 0$ otherwise
- $x = (x_r : r \in R)$ is the vector of flows on routes
- $f = (f_l : l \in L)$ is the vector of flows on links, given by $f = Ax$
- $C = (C_l : l \in L)$ is the vector of link capacities, assumed finite
- $U_r(x_r)$ is the utility function for route r, assumed concave, continuously differentiable, nondecreasing on $[0, +\infty)$.

(For simplicity routing is not considered.) The system optimization problem is $\max_{x:x\geq 0, Ax\leq C} \sum_r U_r(x_r)$. Suppose x^* is feasible ($x^* \geq 0$ and $Ax^* \leq C$) and satisfies the following condition (with $f^* = Ax^*$: There exists a vector $p = (p_l)$ for each link such that

$$\begin{aligned} p_l &\geq 0, && \text{for } l \in L, \text{ with equality if } f_l^* < C_l \\ U'(x_r^*) &\leq \sum_{l\in r} p_l && \text{for } r \in R, \text{ with equality if } x_r^* > 0 \end{aligned}$$

In this problem you are to prove that x^* is a solution to the system optimization problem.
(a) Define the Lagrangian

$$L(x, p) = \sum_r U_r(x_r) + \sum_l p_l \left(C_l - \sum_{r:l\in r} x_r \right)$$

Show that $L(x^*, p) \geq L(x, p)$ for any other vector x with $x \geq 0$. (Hint: There is no capacity constraint, and $L(x, p)$ is concave in x). (b) Deduce from (a) that x^* is a solution to the system optimization problem. (Note: Since the feasible set is compact and the objective function is continuous, a maximizer x^* exits. Since the objective function is continuously differentiable and concave and the constraints linear, a price vector p satsisfying the above conditions also exists, by a standard result in nonlinear optimization theory based on Farkas' lemma.)

8. Fair flow allocation with hard constrained links
Consider four flows in a network of three links as shown in Figure 6. Assume the capacity of each link

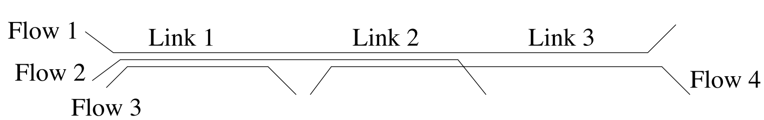

Figure 6: Network with four flows

is one. (a) Find the max-min fair allocation of flows. (b) Find the proportionally fair allocation of flows, assuming the flows are equally weighted. This allocation maximizes $\sum_i \log(x_i)$, where x_i is the rate of the ith flow. Indicate the corresponding link prices.