ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
PROBLEM SET 7 **DUE Wednesday, December 6**

Topic: Dynamic routing, control of queues

1. Illustration of dynamic programming – a stopping time problem
Let $n \geq 1$ and let $X_1, \ldots, X_n$ be mutually independent, exponentially distributed random variables with mean one. A player observes the values of the random variables one at a time. After each observation the player decides whether to stop making observations or to continue making observations. The player's score for the game is the last value observed. Let V_n denote the expected score for an optimal policy. Note that $V_1 = 1$. (a) Express V_{n+1} as a function of V_n. (Hint: Condition on the value of X_1.) (b) Describe the optimal policy for arbitrary value n. (c) For $1 \leq n \leq 5$, compare V_n to $E[\max(X_1, \ldots, X_n)] = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which is the maximum expected score in case the player learns the values of the X's before play begins.

2. Comparison of four ways to share two servers
Let N_i denote the mean total number of customers in system i in equilibrium, where systems 1 through 4 are shown. The arrival process for each system is Poisson with rate 2λ and the servers are exponential with parameter μ. (a) Compute N_i as a function of $\rho = \lambda/\mu$ for $1 \leq i \leq 3$. (Let me know if you can

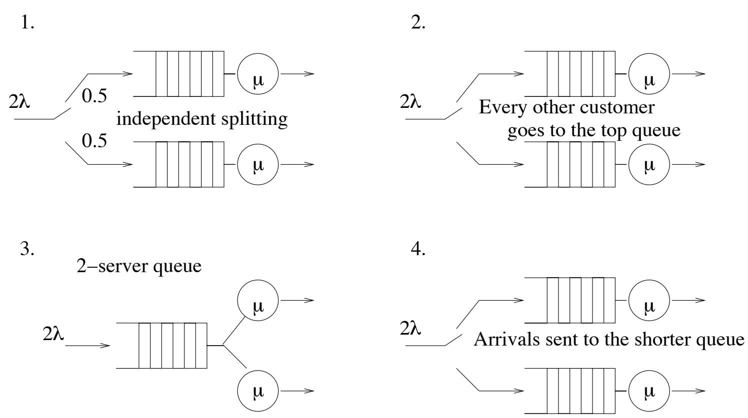

Figure 7: Four two-server systems

find N_4.) (b) Order N_1, N_2, N_3, N_4 from smallest to largest. Justify answer. Does your answer depend on ρ?

3. A stochastic dynamic congestion control problem
Consider a single server queue with finite waiting room so that the system can hold at most K customers. The service times are independent and exponential with parameter μ. If a customer arrives and finds the system full then the customer is rejected and a penalty c is assessed. For each customer that is served a reward r is received at the time of the customer's departure. There are two sources of customers. One is Poisson with fixed rate λ_o and the other is a variable rate source with instantaneous rate λu, where λ is a constant and u is the control value, assumed to lie in the interval [0,1]. The control value u can depend on the state.
(a) Formulate a dynamic programming problem. In particular, describe the state space, the instantaneous cost function, the distribution of inter-event times, and the transition matrix $P(u)$ (draw the transition probability diagram). (b) Give the equations for the cost-to-go functions V_n. (Assume a discount factor $\beta < 1$ for the discrete time formulation as in class.) (c) Describe the optimal control law explicitly in terms of the cost-to-go functions. (d) Speculate about the structure of optimal policy. Can you prove your conjecture? (e) Suppose the assumptions were modified so that customers from the first source are served

at rate μ_1 and that customers from the second are served at rate μ_2. Describe the state space that would be needed for a controlled Markov description of the system under (i) FCFS service order, (ii) preemptive resume priority to customer from the first source, or (iii) processor sharing.

4. Conversion to discrete time with control dependent cost

The continuous time control problem can still be converted to a discrete time control problem if the cost per unit time function g depends on u because, in that case,

$$E_x \int_0^{t_n} g(X(t), u(t)) e^{-\alpha t} dt = \begin{cases} \frac{1-\beta}{\alpha} E_x \sum_{k=0}^{n-1} \beta^k g(X(t_k), w_k) & \text{if } \alpha > 0 \\ \bar{t}_1 E_x \sum_{k=0}^{n} g(X(t_k), w_k) & \text{if } \alpha = 0 \end{cases} \tag{1}$$

where $\beta = \hat{F}(\alpha) = \int_0^\infty e^{-\alpha s} dF(s)$. (a) Prove (1) in case $\alpha > 0$. (The proof for $\alpha = 0$ is similar.) Thus, by ignoring constant factors, we can take the cost for n terms in discrete time to be $E_x \sum_{k=0}^{n-1} g(X(t_k), w_k)$.
(b) How should the fundamental backwards recursion of dynamic programming be modified to take into account the dependence of g on u?

5. A dynamic server rate control problem with switching costs

Consider the following variation of an M/M/1 queueing system. Customers arrive according to a Poisson process with rate λ. The server can be in one of two states–the low state or the high state. In addition, the server is either busy or idle, depending on whether there are customers in the system. A control policy is to be found which determines when the server should switch states, either from low to high or from high to low. The costs involved are a cost per unit time for each customer waiting in the system, a cost per unit time of having the server in the high state, and a switching cost, assessed each time that the server is switched from the low state to the high state. Specifically, let μ_H (respectively, μ_L) denote the service rate when the server is in the high state (respectively, low state), where $\mu_H > \mu_L > 0$. Let c_H denote the added cost per unit time for having the server in the high state, and let c_S denote the cost of switching the server from the low state to the high state. Assume that the control can switch the state of the server only just after a customer arrival or potential departure. Assume there is no charge for switching the server from the high state to the low state. Finally, let c_W denote the cost per unit time per customer waiting in the system. The controller has knowledge of the server state and the number of customers in the system. Suppose the goal is to minimize the expected cost over the infinite horizon, discounted at rate α. (a) Formulate the control problem as a semi-Markov stochastic control problem. In particular, indicate the state space you use, and the transition probabilities. (Hint: Use a control dependent cost function to incorporate the switching cost. The switching cost is incurred at discrete times, in the same way the rewards are given at discrete times for the $M_{\text{controlled}}/M/1$ example in the notes, and can be handled similarly. Using a two dimensional control, one coordinate to be used in case of an arrival, and one coordinate to be used in case of a departure, is helpful.) (b) Let V_n denote the cost-to-go function when there are n steps in the equivalent discrete-time control problem. Write down the dynamic programming equation expressing V_{n+1} in terms of V_n and indicate how the optimal control for n steps to go depends on V_n. (c) Describe the qualitative behavior you expect for the optimal control for the infinite horizon problem. In particular, indicate a particular threshold or switching curve structure you expect the control to have. (d) Describe the optimal control policy when the following two conditions both hold: there is no switching cost (i.e. $c_S = 0$), and $c_W(\mu_H + \alpha) > (\mu_L + \alpha)(c_W + c_H)$. (Hint: Consider the expected cost incurred during the service of one customer.)