ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
SOLUTIONS TO PROBLEM SET 1

1. Poisson merger
Suppose X_i is a Poisson random variable with mean λ_i for $i \in \{1,2\}$ such that X_1 and X_2 are independent. Let $X = X_1 + X_2$, $\lambda = \lambda_1 + \lambda_2$, and $p_i = \lambda_i/\lambda$. Then for $k \geq 0$,
$P[X = k] = \sum_{j=0}^{i} P[X_1 = j]P[X_2 = k-j] = \sum_{j=0}^{i} \frac{e^{-\lambda_1}\lambda_1^i}{i!}\frac{e^{-\lambda_2}\lambda_2^{k-j}}{(k-j)!} = \frac{e^{-\lambda}\lambda^k}{k!}[\sum_{j=0}^{n}\binom{k}{j}p_1^j p_2^{k-j}] = \frac{e^{-\lambda}\lambda^k}{k!}$, where we use the fact that the sum in square brackets is the sum over the binomial distribution with parameters k and p_1. Thus, the sum of two independent Poission random variables is a Poisson random variable. By induction this implies that the sum of any number of independent Poisson random variables is also Poisson.

Now suppose that $N_j = (N_j(t) : t \geq 0)$ is a Poisson process with rate λ_j for $1 \leq j \leq K$, and that $N_1, \ldots, N_K$ are mutually independent. Given $t_0 < t_1 < \cdots < t_p$, consider the following array of random variables:

$$\begin{array}{cccc} N_1(t_1) - N_1(t_0) & N_1(t_2) - N_1(t_1) & \cdots & N_1(t_p) - N_1(t_{p-1}) \\ N_2(t_1) - N_2(t_0) & N_2(t_2) - N_2(t_1) & \cdots & N_2(t_p) - N_2(t_{p-1}) \\ \vdots & \vdots & \cdots & \vdots \\ N_K(t_1) - N_K(t_0) & N_K(t_2) - N_K(t_1) & \cdots & N_K(t_p) - N_K(t_{p-1}) \end{array} \qquad (1)$$

The rows of the array are independent by assumption, and the variables within a row of the array are independent and Poisson distributed by Proposition 1.5.1, characterization (iii). Thus, the elements of the array are mutually independent and each is Poisson distributed. Let $N(t) = N_1(t) + \cdots + N_K(t)$ for all $t \geq 1$. Then the vector of random variables

$$N(t_1) - N(t_0) \quad N(t_2) - N(t_1) \quad \cdots \quad N(t_p) - N(t_{p-1}) \qquad (2)$$

is obtained by summing the rows of the array. Therefore, the variables of the row are independent, and for any i the random variable $N(t_{i+1}) - N(t_i)$ has the $Poi(\lambda(\lambda(t_{i+1} - t_i))$ distribution, where $\lambda = \lambda_1 + \cdots + \lambda_K$. Thus, N is a rate λ Poisson process by Proposition 1.5.1, characterization (iii).

2. Poisson splitting
This is basically the first problem in reverse. Let X be Possion random variable, and let each of X individuals be independently assigned a type, with type i having probability p_i, for some probability distribution $p_1, \ldots, p_K$. Let X_i denote the number assigned type i. Then,

$$\begin{aligned} P(X_1 = i_1, X_2 = i_2, \cdots, X_K = i_K) &= P(X = i_1 + \cdots + i_K)\binom{i_1 + \cdots + i_K}{i_1!\ i_2!\ \cdots\ i_K!} p_1^{k_1} \cdots p_K^{i_K} \\ &= \prod_{j=1}^{K} \frac{e^{-\lambda_j}\lambda_j^{i_j}}{i_j!} \end{aligned}$$

where $\lambda_i = \lambda p_i$. Thus, independent splitting of a Poisson number of individuals yields that the number of each type i is Poisson, with mean $\lambda_i = \lambda p_i$ and they are independent of each other.

Now suppose that N is a rate λ process, and that N_i is the process of type i points, given independent splitting of N with spilt distribution $p_1, \ldots, p_K$. Proposition 1.5.1, characterization (iii). the random variables in (2) are independent, with the i^{th} having the $Poi(\lambda(\lambda(t_{i+1} - t_i))$ distribution. Suppose each column of the array (1) is obtained by independent splitting of the corresponding variable in (2). Then by the splitting property of random variables, we get that all elements of the array (1) are independent, with the appropriate means. By Proposition 1.5.1, characterization (iii), the i^{th} process N_i is a rate λp_i random process for each i, and because of the independence of the rows of the array, the K processes $N_1, \ldots, N_K$ are mutually independent.

3. Poisson method for coupon collector's problem

(a) In view of the previous problem, the number of coupons of a given type i that arrive by time t has the Possion distribution with mean $\frac{t}{k}$, and the numbers of arrivals of different types are independent. Thus, at least one type i coupon arrives with probability $1 - e^{-t/K}$, and $p(k,t) = (1 - e^{-t/k})^k$.

(b) Therefore, $p(k, k\ln k + kc) = (1 - \frac{e^{-c}}{k})^k$, which, by the hint, converges to $e^{-e^{-c}}$ as $k \to \infty$.

(c) The increments of A over intervals of the form $[k, k+1]$ are independent, Poi(1) random variables. Thus, the central limit theorem can be applied to $A(t)$ for large t, yielding that for any constant D, $\lim_{t\to\infty} P[A(t) \geq t + D\sqrt{t}] = Q(D)$, where Q is the complementary normal CDF. The problem has to do with deviations of size $(c - c')k$, which for any fixed D, grow to be larger than $D\sqrt{k\ln k + kc}$. Thus, with $\epsilon = |c - c'|$, $P[|A(k\ln k + kc) - (k\ln k + kc)| \geq \epsilon k] \leq Q(D)$ for k large enough, for any D. Thus, $\lim_{k\to\infty} P[|A(k\ln k + kc) - (k\ln k + kc)| \geq \epsilon k] = 0$, which is equivalent to the required fact.

(d) Let C_t be the event that the collection is complete at time t. Condition on the value of $A(t)$.

$$p(k,t) = P(C_t) = P(C_t|A(t) \geq n)P(A(t) \geq n) + P(C_t|A(t) < n)P(A(t) < n) \tag{3}$$

First we bound from below the right side of (3). Since getting more coupons can only help complete a collection, $P(C_t|A(t) \geq n) \geq P(C_t|A(t) = n) = d(k,n)$, and other terms on the righthand side of (3) are nonnegative. This yields the desired lower bound.

Similarly, to bound above the right side of (3) we use $P(C_t|A(t) \geq n) \leq 1$ and $P(C_t|A(t) < n) \leq d(k,n)$.

(e) Fix c. The first part of the inequality of part (c) yields that $d(k,n) \leq p(k,t)/P[A(t) \geq n]$. Suppose $n = k\ln k + kc$. Let $c' > c$ and take $t = k\ln k + kc'$ as $k \to \infty$. Then $p(k,t) \to e^{-e^{-c'}}$ by part (b) and $P[A(t) \geq n] \to 1$ Thus, $\limsup d(k,n) \leq e^{-e^{-c'}}$. Since c' is arbitrary with $c' > c$, $\limsup d(k,n) \leq e^{-e^{-c}}$. A similar argument shows that $\liminf d(k,n) \geq e^{-e^{-c}}$. Therefore, $\lim d(k,n) = e^{-e^{-c}}$.

4. The sum of a random number of random variables

(a)

$$\begin{aligned} E[S] &= \sum_{n=0}^{\infty} E[S|N=n]P[N=n] = \sum_{n=0}^{\infty} E[X_1 + \cdots + X_n|N=n]P[N=n] \\ &= \sum_{n=0}^{\infty} E[X_1 + \cdots + X_n]P[N=n] = \sum_{n=0}^{\infty} n\overline{X}P[N=n] = \overline{X}\ \overline{N}. \end{aligned}$$

Similarly,

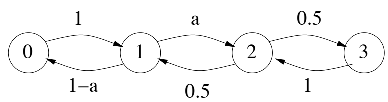
$$\begin{aligned} E[S^2] &= \sum_{n=0}^{\infty} E[(X_1 + \cdots + X_n)^2]P[N=n] \\ &= \sum_{n=0}^{\infty} \left[E[X_1 + \cdots + X_n]^2 + \mathrm{Var}(X_1 + \cdots + X_n)\right] P[N=n] \\ &= \sum_{n=0}^{\infty} \left[(n\overline{X})^2 + n\mathrm{Var}(X)\right] P[N=n] \\ &= \overline{N^2}(\overline{X})^2 + \overline{N}\mathrm{Var}(X) \end{aligned}$$

so that $\mathrm{Var}(S) = E[S^2] - Es^2 = \mathrm{Var}(N)(\overline{X})^2 + \overline{N}\mathrm{Var}(X)$.

(b) By the same reasoning as in part (a),

$$\begin{aligned} \Phi_S(u) &= E[e^{juS}] = \sum_{n=0}^{\infty} E[e^{ju(X_1 + \cdots + X_n)}]P[N=n] \\ &= \sum_{n=0}^{\infty} \Phi_{X_1}(u)^n P[N=n] = B(\Phi_{X_1}(u)) \end{aligned}$$

5. Mean hitting time for a simple Markov process
(a)

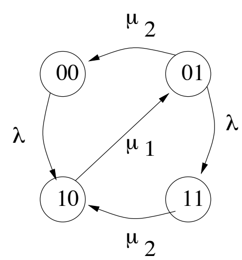


Solve $\pi = \pi P$ and $\pi e = 1$ to get $\pi = \left(\frac{1-a}{2(1+a)}, \frac{1}{2(1+a)}, \frac{2a}{2(1+a)}, \frac{a}{2(1+a)}\right)$ for all $a \in [0,1]$ (unique).
(b) A general way to solve this is to let $h_i = E[\min\{n \geq 0 | X(n) = 3\} | X(0) = i]$, for $0 \leq i \leq 3$. Our goal is to find h_0. Trivially, $h_3 = 0$. Derive equations for the other values by conditioning on the first step of the process: $h_i = 1 + \sum_j p_{ij} h_j$ for $i \neq 3$. Or

$$\begin{aligned} h_0 &= 1 + h_1 \\ h_2 &= 1 + (1-a)h_0 + ah_2 \\ h_2 &= 1 + (0.5)h_i \end{aligned} \quad \text{yielding} \quad \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{a} + 1 \\ \frac{4}{a} \\ \frac{2}{a} + 1 \end{pmatrix}.$$

Thus, $h_0 = \frac{4}{a} + 1$ is the required answer.

6. A two station pipeline in continuous time
(a) $\mathcal{S} = \{00, 01, 10, 11\}$
(b)



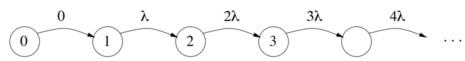
(c) $Q = \begin{pmatrix} -\lambda & 0 & \lambda & 0 \\ \mu_2 & -\mu_2 - \lambda & 0 & \lambda \\ 0 & \mu_1 & -\mu_1 & 0 \\ 0 & 0 & \mu_2 & -\mu_2 \end{pmatrix}$.
(d) $\eta = (\pi_{00} + \pi_{01})\lambda = (\pi_{01} + \pi_{11})\mu_2 = \pi_{10}\mu_1$. If $\lambda = \mu_1 = \mu_2 = 1.0$ then $\pi = (0.2, 0.2, 0.4, 0.2)$ and $\eta = 0.4$.
(e) Let $\tau = \min\{t \geq 0 : X(t) = 00\}$, and define $h_s = E[\tau | X(0) = s]$, for $s \in \mathcal{S}$. We wish to find h_{11}.

$$\begin{aligned} h_{00} &= 0 \\ h_{01} &= \frac{1}{\mu_2+\lambda} + \frac{\mu_2 h_{00}}{\mu_2+\lambda} + \frac{\lambda h_{11}}{\mu_2+\lambda} \\ h_{10} &= \frac{1}{\mu_1} + h_{01} \\ h_{11} &= \frac{1}{\mu_2} + h_{10} \end{aligned}$$

For If $\lambda = \mu_1 = \mu_2 = 1.0$ this yields $\begin{pmatrix} h_{00} \\ h_{01} \\ h_{10} \\ h_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 5 \end{pmatrix}$. Thus, $h_{11} = 5$ is the required answer.

7. Simple population growth models

(a)



(b) $\pi_0(t) \equiv 0$. The Kolmogorov forward equations are given, for $k \geq 1$, by
$\frac{\partial \pi_k}{\partial t} = (k-1)\lambda\pi_{k-1}(t) - k\lambda\pi_k(t)$.
Multiplying each side by λ^k and summing yields

$$\begin{aligned} \frac{\partial P(z,t)}{\partial t} &= \sum_{k=1}^{\infty} \frac{\partial \pi_k(t)}{\partial t} z^k = \lambda \left[z^2 \sum_{k=2}^{\infty} \pi_{k-1}(t)(k-1)z^{k-1} - z \sum_{k=1}^{\infty} \pi_k(t) k z^{k-1} \right] \\ &= \lambda(z^2 - z)\frac{\partial P(z,t)}{\partial z} \end{aligned}$$

That is, $P(z,t)$ is a solution of the partial differential equation:

$$\frac{\partial P(z,t)}{\partial t} = \lambda(z^2 - z)\frac{\partial P(z,t)}{\partial z}$$

with the initial condition $P(z,0) = z$. It is easy to check that the expression given in the problem statement is a solution. Although it wasn't requested in the problem, here is a method to find the solution. The above PDE is a first order linear hyperbolic (i.e. wave type) equation which is well-posed and is readily solved by the method of characteristics. The idea is to rewrite the equation as $[\frac{\partial}{\partial t} - \lambda(z^2 - z)\frac{\partial}{\partial z}]P(a,t) = 0$, which means that $P(z,t)$ has directional derivative zero in the direction $(-\lambda(z - z^2), 1)$ in the (z,t) plane.
(c) Write $\overline{N}_t = E[N(t)]$. We could use the expression for $P(z,t)$ given and compute $\overline{N}_t = \frac{\partial P(z,t)}{\partial z}|z = 0$. Alternatively, divide each side of the PDE by $z-1$. Then since $\frac{\partial(1)}{\partial t} = 0$, we get $\frac{\partial}{\partial t}\left[\frac{P(z,t)-1}{z-1}\right] = \lambda z \frac{\partial P(z,t)}{\partial z}$. Letting $z \to 1$ then yields the differential equation $\frac{d\overline{N}_t}{dt} = \lambda \overline{N}_t$, so that $\overline{N}_t = e^{\lambda t}$.
(d) By part (b), $P(z,t) = \frac{ze^{-\lambda t}}{1 - z(1 - e^{-1\lambda t})} = \sum_{k=1}^{\infty} e^{-\lambda t}(1 - e^{-\lambda t})^{k-1} z^k$ so that $\pi_k(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}$, for $k \geq 1$. That is, the population size at time t has the geometric distribution with parameter $e^{-\lambda t}$.
(e) For the deterministic model, $n_t = 2^{\lfloor \lambda t \rfloor} \approx 2^{\lambda t} = e^{(\ln 2)\lambda t}$. Thus, the growth rate for the deterministic model is roughly exponential with exponent $(\ln 2)\lambda \approx (0.693)\lambda$, which is considerably smaller than the exponent λ for the random model.

8. Equilibrium distribution of the jump chain

The choice of B insures that $(-Bq_{ii}\pi_i)_{i \in \mathcal{S}}$ is a probability distribution. Since $p_{ij}^J = \begin{cases} 0 & \text{if } i = j \\ \frac{q_{ij}}{-q_{ii}} & \text{if } i \neq j \end{cases}$, we have for fixed j,

$$\begin{aligned} \sum_{i \in \mathcal{S}} (-Bq_{ii}\pi_i) p_{ij}^J &= \sum_{i:i \neq j} -Bq_{ii}\pi_i \left(\frac{q_{ij}}{-q_{ii}} \right) \\ &= B \sum_{i:i \neq j} \pi_i q_{ij} = -B\pi_j q_{jj} \end{aligned}$$

Thus, $-B\pi_i q_{ii}$ is the equilibrium vector for P^J. In other words, $-B\pi_i q_{ii} = \pi_i^J$.
Another justification of the same result goes as follows. For the original Markov process, $-q_{ii}^{-1}$ is the mean holding time for each visit to state i. Since π_i^J represents the proportion of jumps of the process which land in state i, we must have $\pi_i \propto \pi_i^J(-q_{ii})^{-1}$, or equivalently, $\pi_i^J \propto (-q_{ii})\pi_i^J$.