ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
SOLUTIONS TO PROBLEM SET 2

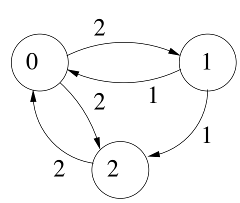
1. A simple Poisson process calculation
Suppose $0 < s < t$ and $0 \leq i \leq k$.

$$\begin{aligned} P[N(s) = i | N(t) = k] &= \frac{P[N(s) = i, N(t) = k]}{P[N(t) = k]} \\ &= \left(\frac{e^{-\lambda s}(\lambda s)^i}{i!}\right)\left(\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{k-i}}{(k-i)!}\right)\left(\frac{e^{-\lambda t}(\lambda t)^k}{k!}\right)^{-1} \\ &= \binom{k}{i}\left(\frac{s}{t}\right)^i\left(\frac{t-s}{t}\right)^{k-1} \end{aligned}$$

That is, given $N(t) = k$, the conditional distribution of $N(s)$ is binomial. This could have been deduced with no calculation, using the fact that given $N(t) = k$, the locations of the k points are uniformly and independently distributed on the interval $[0, t]$.

2. An alternating renewal process
(a) Sampled periods between light changes L are deterministic, length one, so the forward residual lifetime γ is uniformly distributed over $[0, 1]$. Thus, $P[\text{light changes}] = P[\gamma \leq 0.5] = 0.5$.
(b) By symmetry, γ is independent of what color is spotted first. If green is spotted first,
$W = \begin{cases} 0 & \text{if } \gamma > 0.5 \\ \gamma + 0.5 & \text{if } \gamma < 0.5 \end{cases}$ so that $E[W|\text{green spotted}] = \int_0^{0.5}(u+0.5)du = \frac{3}{8}$. Similarly, $E[W|\text{red spotted}] = \int_{0.5}^{1}(u - 0.5)du = \frac{1}{8}$. Thus, $E[W] = \frac{3}{8}\frac{1}{2} + \frac{1}{8}\frac{1}{2} = \frac{1}{4}$. This make sense, because the light is red with probability one half when the light is reached, and given it is red when reached, the average wait is 0.5.
(c) $P[\text{light changes}] = P[\gamma \leq 0.5] = \int_0^{0.5} \frac{1-F_X(y)}{m_1}dy = \int_0^{0.5} \frac{1-\frac{y}{2}}{1}dy = \frac{7}{16}$.
(d) 2 blocks/minute (The point here is to express the answer is units given. If the length of a block is specified, the speed could be given in meters/second or miles per hour, for example.)

3. A mean hitting time problem
(a)



$\pi Q = 0$ implies $\pi = (\frac{2}{7}, \frac{2}{7}, \frac{3}{7})$.
(b) Clearly $a_1 = 0$. Condition on the first step. The initial holding time in state i has mean $-\frac{1}{q_{ii}}$ and the next state is j with probability $p^J_{ij} = \frac{-q_{ij}}{q_{ii}}$. Thus $\begin{pmatrix} a_0 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{q_{00}} \\ -\frac{1}{q_{22}} \end{pmatrix} + \begin{pmatrix} 0 & p^J_{02} \\ p^J_{20} & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_2 \end{pmatrix}$. Solving yields $\begin{pmatrix} a_0 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$.
(c) Clearly $\alpha_2(t) = 0$ for all t.

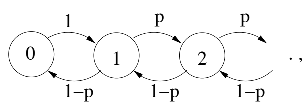
$$\begin{aligned} \alpha_0(t+h) &= \alpha_0(t)(1 + q_{00}h) + \alpha_1(t)q_{10}h + o(h) \\ \alpha_1(t+h) &= \alpha_0(t)q_{01}h + \alpha_1(t)(1 + q_{11}h) + o(h) \end{aligned}$$

Subtract $\alpha_i(t)$ from each side and let $h \to 0$ to yield $(\frac{\partial \alpha_0}{\partial t}, \frac{\partial \alpha_1}{\partial t}) = (\alpha_0, \alpha_1) \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$ with the inital condition $(\alpha_0(0), \alpha_1(0)) = (1, 0)$. (Note: the matrix involved here is the Q matrix with the row and column for state 2 removed.)
(d) Similarly,

$$\begin{aligned} \beta_0(t-h) &= (1+q_{00}h)\beta_0(t) + q_{01}h\beta_1(t) + o(h) \\ \beta_1(t-h) &= q_{10}h\beta_0(t) + (1+q_{11}h)\beta_1(t)) + o(h) \end{aligned}$$

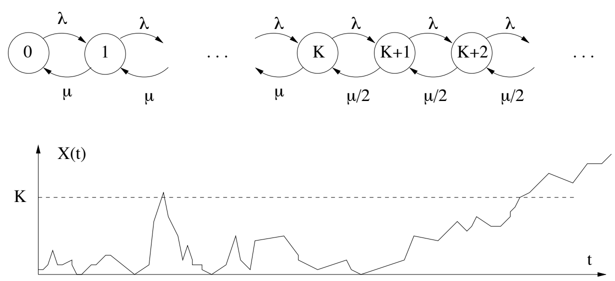
Subtract $\beta_i(t)$'s, divide by h and let $h \to 0$ to get: $\begin{pmatrix} -\frac{\partial \beta_0}{\partial t} \\ -\frac{\partial \beta_1}{\partial t} \end{pmatrix} = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ with $\begin{pmatrix} \beta_0(t_f) \\ \beta_1(t_f) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

4. An unusual birth-death process
(a) $S_2 = \sum_{n=0}^{\infty} \frac{\mu_1 \ldots \mu_n}{\lambda_1 \ldots \lambda_n} = \sum_{n=0}^{\infty} (\frac{1-p}{p})^n < +\infty$ (because $p > 0.5$) so X is transient.
(b) $aQ = 0$ is equivalent to $\lambda_k a_k = \mu_k a_{k+1}$ for $k \geq 0$, which is easily checked.
(c) $p^J_{01} = 1$ and, for $i \geq 1$, $p^J_{ii-1} = 1 - p$ and $p^J_{ii+1} = p$. All other transition probabilities are zero.



(d) S_2 is the same as in (a), so X^J is transient (as is also obvious from part (c).)
Note: X reaches the graveyard state $\triangle$ in finite time. Otherwise, the fact that $Qa = 0$ for a probability distribution a would imply that all states are positive recurrent.

5. A queue with decreasing service rate
(a)

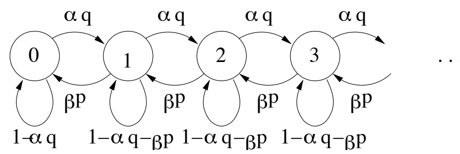


(b) $S_2 = \sum_{k=0}^{\infty} (\frac{\mu}{2\lambda})^k 2^{k \wedge K}$, where $k \wedge K = \min\{k, K\}$. Thus, if $\lambda < \frac{\mu}{2}$ then $S_2 < +\infty$ and the process is recurrent. $S_1 = \sum_{k=0}^{\infty} (\frac{2\lambda}{\mu})^k 2^{-k \wedge K}$, so if $\lambda < \frac{\mu}{2}$ then $S_1 < +\infty$ and the process is positive recurrent. In this case, $\pi_k = (\frac{2\lambda}{\mu}) 2^{-k \wedge K} \pi_0$, where

$$\pi_0 = \frac{1}{S_1} = \left[\frac{1 - (\lambda/\mu)^K}{1 - (\lambda/\mu)} + \frac{(\lambda/\mu)^K}{1 - (2\lambda/\mu)} \right]^{-1}.$$

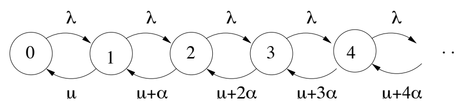
(c) If $\lambda = \frac{2\mu}{3}$, the queue appears to be stable until if fluctuates above K. Eventually the queuelength will grow to infinity at rate $\lambda - \frac{\mu}{2} = \frac{\mu}{6}$. See figure above.

6. Limit of a distrete time queueing system
(q) The transition rate diagram for the number in the system is shown:



$a_k = P$[no arrivals for $k-1$ slots, then arrival] $= (1-\alpha q)^{k-1}\alpha q$ for $k \geq 1$. Similarly, $b_k = (1-\beta q)^{k-1}\beta q$ for $k \geq 1$. Thus, an interarrival time is q times a geometric random variable with parameter αq, and the mean interarrival time is $\frac{1}{\alpha}$. and a service time is q times a geometric random variable with parameter βq, and the mean service time is $\frac{1}{\beta}$.
(b) $\beta q p_{k+1} = \alpha q p_k$ for $k \geq 0$, impies that $p_k = (1-\rho)\rho^k$, where $\rho = \frac{\alpha}{\beta}$. This distribution is the same for all q, so it trivially converges as $q \to 0$ to the same value: $p_k \to (1-\rho)\rho^k$. The interarrival times and service times become exponentially distributed in the limit. The limit system is a continuous time M/M/1 system.

7. An M/M/1 queue with impatient customers
(a)

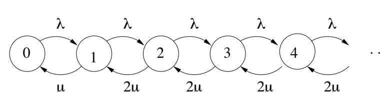


(b) The process is positive recurrent for all λ, μ if $\alpha > 0$, and $p_k = \frac{c\lambda^k}{\mu(\mu+\alpha)\cdots(\mu+(k-1)\alpha)}$ where c is chosen so that the p_k's sum to one.
(c) If $\alpha = \mu$, $p_k = \frac{c\lambda^k}{k!\mu^k} = \frac{c\rho^k}{k!}$. Therefore, $(p_k : k \geq 0)$ is the Poisson distribution with mean ρ. Furthermore, p_D is the mean departure rate by defecting customers, divided by the mean arrival rate λ. Thus,

$$p_D = \frac{1}{\lambda}\sum_{k=1}^{\infty} p_k(k-1)\alpha = \frac{\rho - 1 + e^{-\rho}}{\rho} \to \begin{cases} 1 & \text{as } \rho \to \infty \\ 0 & \text{as } \rho \to 0 \end{cases}$$

where l'Hôspital's rule can be used to find the limit as $\rho \to 0$.

8. Statistical multiplexing
(a) Use an M/M/1 model, with $\lambda = 40$ packets/second, and $\mu = 50$ packets per second. Then $T = \frac{1}{\lambda-\mu} = 0.1$ seconds.
(b) Model each link as an M/M/1 queue, with $\lambda = 20$ packets/second, and $\mu = 25$ packets per second. Then $T = \frac{1}{\lambda-\mu} = 0.2$ seconds.
(c) Model as an M/M/2 queue with $\lambda = 40$ packets/second and $\mu = 25$ packets/second.



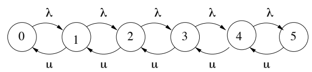
Let $\rho = \frac{\lambda}{2\mu}$. Then $\pi_0 = \frac{0.5}{0.5+\rho+\rho^2+\cdots}$ and $\pi_k = 2\pi_0\rho^k$ for $k \geq 1$. Thus

$$T = \frac{\overline{N}}{\lambda} = \frac{\sum_{k=1}^{\infty} k\rho^k}{\lambda(0.5+\rho+\rho^2+\cdots)} = \frac{\rho/(1-\rho)^2}{\lambda((\frac{1}{1-\rho})-0.5)} = 0.111 \text{ seconds}$$

We can view model (c) as a variation of (b) with statistical multiplexing at the packet level, and (a) as a variation with multiplexing at the subpacket level. Here (c) is a substantial improvement over (b), and (a) is a little better than (c).

9. A queue with blocking

(a)



$\pi_k = \frac{\rho^k}{1+\rho+\rho^2+\rho^3+\rho^4+\rho^5} = \frac{\rho^k(1-\rho)}{1-\rho^6}$ for $0 \le k \le 5$.

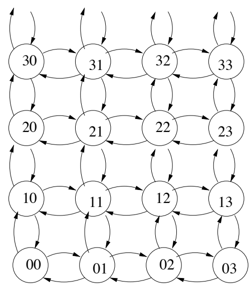
(b) $p_B = \pi_5$ by the PASTA property.

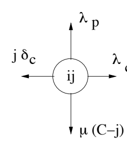
(c) $\overline{W} = \overline{N_W}/(\lambda(1-p_B))$ where $\overline{N_W} = \sum_{k=0}^{5}(k-1)\pi_k$. Alternatively, $\overline{W} = \overline{N}/(\lambda(1-p_B)) - \frac{1}{\mu}$ (i.e. $\overline{W}$ is equal to the mean time in system minus the mean time in service)

(d) $\pi_0 = \frac{1}{\lambda(\text{mean cycle time for visits to state zero})} = \frac{1}{\lambda(1/\lambda + \text{mean busy period duration})}$ Therefore, the mean busy period duration is given by $\frac{1}{\lambda}[\frac{1}{\pi_0} - 1] = \frac{\rho-\rho^6}{\lambda(1-\rho)} = \frac{\rho-\rho^5}{\mu(1-\rho)}$

10. Multiplexing circuit and packet data streams

(a) A portion of the rate diagram for $C = 3$ is pictured.

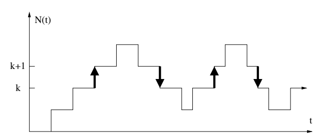




(b) Yes, the phase structure is apparent.

(c) $\lambda_p \le \mu(C - \overline{n}_C) = \frac{\mu\sum_{j=0}^{C}(c-j)(\frac{\lambda_c}{\delta_C})^j/j!}{\sum_{j=0}^{C}(\frac{\lambda_c}{\delta_C})^j/j!}$

11. On two distibutions seen by customers



As can be seen in the picture, between any two transtions from state k to $k+1$ there is a transition form state $k+1$ to k, and vice versa. Thus, the number of transitions of one type is within one of the number of transitions of the other type. This establishes that $|D(k,t) - R(k,t)| \le 1$ for all k.

(b)Observe that

$$\begin{aligned}
\left|\frac{D(k,t)}{\alpha_t} - \frac{R(k,t)}{\delta_t}\right| &\le \left|\frac{D(k,t)}{\alpha_t} - \frac{R(k,t)}{\alpha_t}\right| + \left|\frac{R(k,t)}{\alpha_t} - \frac{R(k,t)}{\delta_t}\right| \\
&\le \frac{1}{\alpha_t} + \frac{R(k,t)}{\alpha_t}\left|1 - \frac{\alpha_t}{\delta_t}\right| \\
&\le \frac{1}{\alpha_t} + \left|1 - \frac{\alpha_t}{\delta_t}\right| \to 0 \quad \text{as} \quad t \to \infty
\end{aligned}$$

Thus, $\frac{D(k,t)}{\alpha_t}$ and $\frac{R(k,t)}{\delta_t}$ have the same limits, if the limits of either exists.