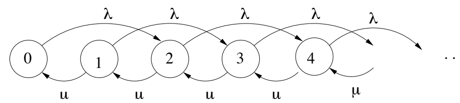
ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
SOLUTIONS TO PROBLEM SET 4

1. A queue with customers arriving in pairs
(a)



(b) For any function V on $\mathbb{Z}$, $QV(x) = \lambda(Q(x+2)-Q(x))+\mu(Q((x-1)_+)-Q(x))$. In particular, if $V(x) = x$, then $QV(x) = -(\mu - 2\lambda) + \mu I_{\{x=0\}}$. Therefore, by the Foster-Lypunov stability criteria, if $\mu \geq 2\lambda$, then N is recurrent, and if $\mu > 2\lambda$, then N is positive recurrent.
(c) Take $V(x) = \frac{x^2}{2}$. Then,

$$QV(x) = \begin{cases} -x(\mu - 2\lambda) + \frac{4\lambda+\mu}{2} & \text{if } x \geq 1 \\ 2\lambda & \text{if } x = 0 \end{cases}$$

Thus $QV(x) \leq -x(\mu - 2\lambda) + \frac{4\lambda+\mu}{2}$ for all $x \in \mathbb{Z}$. Conclude by the combined Foster-Lyapunov criteria and moment bounds that if $\mu > 2\lambda$, then N is positive recurrent, and $\overline{N} \leq \frac{4\lambda+\mu}{2(\mu-2\lambda)}$.
(d) Since the process is not explosive, it is positive recurrent if and only if there is a probability distribution p so that $pQ = 0$. The equation $pQ = 0$ means that the net probability flux into each state should be zero. Equivalently, $pQ = 0$ means that the next flux out of the set $\{0, 1, \ldots, k\}$ is zero for each $k \geq 0$. Defining $p_{-1} = 0$ for convenience, the equation $pQ = 0$ is equivalent to the set of equations: $(p_{k-1} + p_k)\lambda = p_{k+1}\mu$ for $k \geq 0$. (This is a set of second order linear difference equations.) Multiply each side of the k^{th} equation by z^{k+1}, sum from $k = 0$ to ∞ to yield $(z^2\lambda + z\lambda)P(z) = \mu(P(z) - p_o)$, so that

$$P(z) = \frac{p_0}{1 - (\lambda z/\mu) - (\lambda z^2/\mu)}$$

The condition $P(1) = 1$ yields that $p_0 = 1 - 2\lambda/\mu$, which is valid only if $2\lambda/\mu < 1$. Under that condition,

$$P(z) = \frac{1 - 2\lambda/\mu}{1 - (\lambda z/\mu) - (\lambda z^2/\mu)}$$

The method of partial fraction expansion can be used to express P as the sum of two terms with degree one denominators. First, P is rewritten as

$$P(z) = \frac{1 - 2\lambda/\mu}{(1 - z/a)(1 - z/b)}$$

where a and b are the poles of P (i.e. the zeros of the denominator), given by

$$a = \frac{-1 + \sqrt{1 + \frac{4\mu}{\lambda}}}{2} \quad b = \frac{-1 - \sqrt{1 + \frac{4\mu}{\lambda}}}{2}$$

Matching P and its partial fraction expansion near the poles yields

$$P(z) = (1 - 2\lambda/\mu)\left[\frac{1}{(1 - z/a)(1 - a/b)} + \frac{1}{(1 - b/a)(1 - z/b)}\right]$$

so that

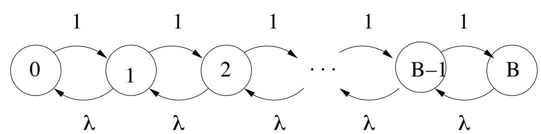
$$p_k = (1 - 2\lambda/\mu)\left[\frac{a^{-k}}{1 - a/b} + \frac{b^{-k}}{1 - b/a}\right] \qquad k \geq 0$$

(Using this expression we can also find an explicit expression for $\overline{N}$, but it is more complicated than the bound on $\overline{N}$ found in part (c).) Thus, we have found that there exists a probability distribution p such that $pQ = 0$ if and only if $2\lambda/\mu < 1$.
(e) $r_k = p_k$ by the PASTA property. d_k is proportional to $p_{k+1}\mu$ for $k \geq 1$, so that $d_k = \frac{p_{k+1}}{1-p_0}$.

2. Token bucket regulation of a Poisson stream
Suppose for simplicity that the tokens arrive at time $0, 1, 2, \cdots$, and let X_k denote the number of tokens present at time $k-$ (Another approach would be to consider the number of tokens just after time k.) If $\beta_k = \frac{e^{-\lambda}\lambda^k}{k!}$, then the one-step transition probability matrix for $X = (X_k : k \geq 0)$ is

$$P = \begin{pmatrix} 1-\beta_0 & \beta_0 & & & & \\ 1-\beta_0-\beta_1 & \beta_1 & \beta_0 & & & \\ 1-\beta_0-\beta_1-\beta_2 & \beta_2 & \beta_1 & \beta_0 & & \\ \vdots & \vdots & & \ddots & & \\ 1-\beta_0-\cdots-\beta_{B-1} & & & & \beta_1 & \beta_0 \\ 1-\beta_0-\cdots-\beta_{B-1} & & & & \beta_1 & \beta_0 \end{pmatrix}$$

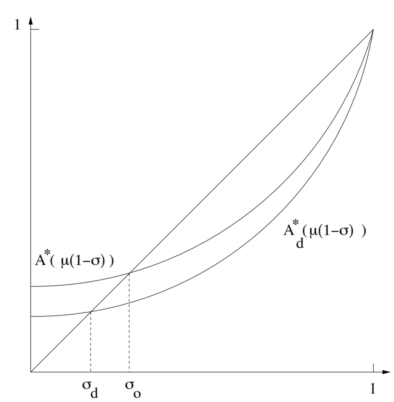
(b) The throughput of packets is equal to the throughput of tokens, where is $1 - \pi_B$. Therefore, $P_{loss\ for\ packets} = \frac{\lambda-(1-\pi_B)}{\lambda}$. (The expression used here would be different for the other choice of X mentioned in the solution to part (a)).
(c) Let $Z(t)$ denote the number of tokens present at time t. Then Z is a continuous time Markov process with the transition rate diagram shown.



Then,

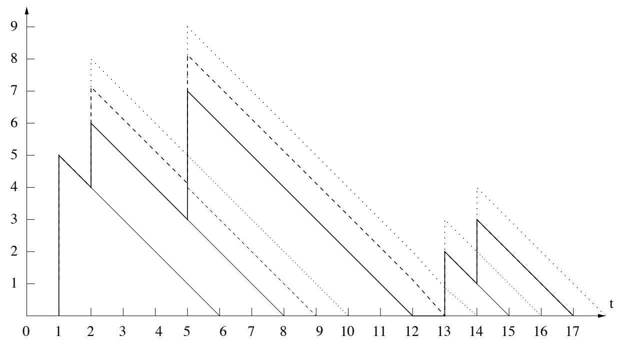
$$P_{loss\ for\ packets} = P[\text{no tokens present}] = \frac{1}{1+\frac{1}{\lambda}+\cdots+\frac{1}{\lambda^B}} = \frac{\lambda^B}{\lambda^B+\lambda^{B-1}+\cdots+1}$$

3. Extremality of constant interarrival times for G/M/1 queues
(a) $A_d^*(s) = \exp(-s/\lambda)$. Let t_1 have distribution function A. Then since $\exp(-sx)$ is convex in x, Jensens inequality can be applied to yield $A^*(s) = E[\exp(-st_1)] \geq \exp(-sE[t_1]) = A_d^*(s)$.
(b) Let σ_o and σ_d be values of σ for the respective queues. Since $A_d^*(\mu(1-\sigma_o)) \leq A^*(\mu(1-\sigma_0) = \sigma_0$ and A_d^* is convex, it follows (see the figure) that $\sigma_d \leq \sigma_o$. The assertions follow because the required means are increasing functions of σ.



4. Propagation of perturbations

(a) See the solid lines in the figure. The departure times are indicated by where the solid diagonal lines meet the axis, and occur at times 6,8,12,15, and 17. The waiting times in queue are 0, 4, 3, 0, 1.



(b) The departure times are indicated by where the dashed diagonal lines meet the axis, and occur at times 6,9,13,15, and 17. The waiting times in queue are 0, 4, 4, 0, 1.
(c) The departure times are indicated by where the dashed diagonal lines meet the axis, and occur at times 6,10, 14, 16, and 18. The waiting times in queue are 0, 4, 5, 1, 2.
(d) The waiting times of all following customers in the same busy period increase by the same amount. Busy periods can merge, and the waiting times of customers originally in a later busy period can also increase.

5. On priority M/GI/1 queues

(a) The server works on a type 1 customer whenever one is present, and otherwise works on a type 2 customer, if one is present. If the server is interrupted by a type 1 customer when it is working on a type 2 customer, the service of the type 2 customer is suspended until no more type 1 customers are in the system. Then service of the type 2 customer is resumed, with its remaining service time the same as when the service was interrupted.

(b) Let $\lambda = \lambda_1 + \lambda_2$, $\rho_i = \lambda_i \overline{X_i}$, and $\rho = \rho_1 + \rho_2$. Then $\overline{X} = \frac{\sum_i \lambda_i \overline{X_i}}{\lambda}$ and $\overline{X^2} = \frac{\sum_i \lambda_i \overline{X_i^2}}{\lambda}$ Type 1 customers are not effected by type 2 customers, so the mean system time for type 1 customers is the same as if there were no type 2 customers:

$$T_1 = W_1 + \overline{X_1} = \frac{\lambda_1 \overline{X_!^2}}{2(1-\rho_1)} + \overline{X_1}$$

where $\rho_1 = \lambda_1 \overline{X_1}$. To find an expression for T_2, consider a type 2 customer. Recall that the mean work in the system is given by $\frac{\lambda \overline{X^2}}{2(1-\rho)}$. The PASTA property holds. Thus, T_2 is the mean of the sum of the work already in the system when the customer arrives, the service time of the customer itself, and the amount of preempting work that arrives while the type 2 customer is in the system:

$$T_2 = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \overline{X_2} + \rho_1 T_2.$$

Solving for T_2 yields

$$T_2 = \left(\frac{1}{1-\rho_1}\right)\left(\frac{\lambda \overline{X^2}}{2(1-\rho)} + \overline{X_2}\right)$$

6. Optimality of the μc rule
Let $J(\sigma) = \sum_i \lambda_i c_i W_i(\sigma) = \sum_i \mu_i c_i \rho_i W_i(\sigma)$ denote the cost for permutatation σ.

Claim: Let σ be a permutation and let $\hat{\sigma}$ denote another permutation obtained from σ by swapping σ_i and σ_{i+1} for some i. Then $J(\hat{\sigma}) < J(\sigma)$ if and only if $\mu_{\sigma_i} c_{\sigma_i} < \mu_{\sigma_{i+1}} c_{\sigma_{i+1}}$.

The claim is proved as follows. Since only customers of classes i and $i+1$ are affected by the swap, $W_j(\hat{\sigma}) = W_j(\sigma)$ for $j \notin \{\sigma_i, \sigma_{i+1}\}$. On the other hand, since class σ_i (i.e. class $\hat{\sigma}_{i+1}$) customers have lower priority under $\hat{\sigma}$ than under σ, $W_{\sigma_i}(\hat{\sigma}) > W_{\sigma_i}(\sigma)$. Thus, $\delta > 0$, where

$$\delta = \rho_{\sigma_i} W_{\sigma_i}(\hat{\sigma}) - \rho_{\sigma_i} W_{\sigma_i}(\sigma).$$

Similarly, $W_{\sigma_{i+1}}(\hat{\sigma}) < W_{\sigma_{i+1}}(\sigma)$, and furthermore by the conservation of flow equations,

$$\rho_{\sigma_{i+1}} W_{\sigma_{i+1}}(\hat{\sigma}) - \rho_{\sigma_{i+1}} W_{\sigma_{i+1}}(\sigma) = -\delta.$$

Therefore, $J(\hat{\sigma}) - J(\sigma) = (\mu_{\sigma_i} c_{\sigma_i} - \mu_{\sigma_{i+1}} c_{\sigma_{i+1}})\delta$. Since $\delta > 0$, the claim follows immediately.

By the claim, if σ does not satisfy the ordering condition $\mu_{\sigma_1} c_{\sigma_1} \geq \mu_{\sigma_2} c_{\sigma_2} \geq \ldots \geq \mu_{\sigma_K} c_{\sigma_K}$, then σ is not optimal.

Conversely, if σ does satisfy the ordering condition and if σ^0 is an arbitrary ordering, there is a sequence of orderings $\sigma^0, \sigma^1, \ldots, \sigma^p = \sigma$ so that $J(\sigma^0) \geq J(\sigma^1) \geq \cdots \geq J(\sigma^p) = J(\sigma)$, so that σ is optimal.

The μc rule is intuitively plausible because μc is the rate that instantaneous cost is reduced when a rate μ server works on a customer with instantaneous holding cost c.

7. A discrete time M/GI/1 queue
The mean service time is 7/2, so the service rate is 2/7. Let $\rho = 7p/2$, which is the arrival rate, p, divided by the service rate. Also, ρ is the probability the server is busy in a time slot. Arrivals see the system in equilibrium by the discrete time version of the PASTA property, which is true when numbers of arrivals in distinct slots are independent. Given the server is busy during the slot of an arrival, the residual service time γ of the customer in service is distributed over $\{0, 1, 2, 3, 4, 5\}$. The total service time of the customer in service has distribution $P[l = l] = \frac{f_l}{m_1}$, for $1 \leq l \leq 6$, with mean $E[L] = \frac{m_2}{m_1} = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6m_1} = \frac{13}{3}$. Given L, γ is uniform on $\{0, 1, \ldots, L-1\}$, so $E[\gamma] = \frac{E[L]-1}{2} = \frac{5}{3}$. Thus, $W = 3.5N + (\frac{7p}{2})(\frac{5}{3})$, where N is the mean number of customers seen by an arrival, equal to pW. Solving for W yields $W = \frac{35p}{6-21p}$ for $0 \leq p < \frac{2}{7}$. An embedded Markov process is given by the number of customers in the system just after the departure of a customer. To be specific, this number could include the new arrival, if any, occurring is the same slot that a service is completed.

Here ia a second way to solve the problem, obtained by viewing the events at slot boundaries in a different order. Suppose that the customer in service exits if its service is complete, and if so, the customer at the head of the queue (if any) is moved into the server before a possible arrival is added. In this view, a new customer still arrives to find the server busy with probability $\rho = \frac{7p}{2}$, but the residual service time is distributed over $\{1, 2, 3, 4, 5, 6\}$. The total service time of the customer in service hs the same sampled lifetime distribution as before , with mean $E[L] = \frac{13}{3}$, but here, given L, γ is uniformly distributed over $\{1, \ldots, L\}$, so $E[\gamma] = \frac{8}{3}$. Thus, by the PASTA property, $W = 3.5N + (\frac{7p}{2})(\frac{8}{3})$, where N is the mean number in the queue seen by an arrival. A customer waiting W slots for service can be seen in the queue for $W - 1$ observation times, so by Little's law, $N = (W-1)p$. thus, $W = 3.5(W-1)p + (\frac{7p}{2})(\frac{8}{3})$, and solving for W yields $W = \frac{35p}{6-21p}$ for $0 \leq p < \frac{2}{7}$.