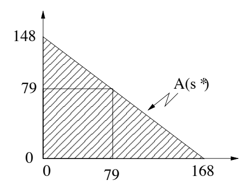
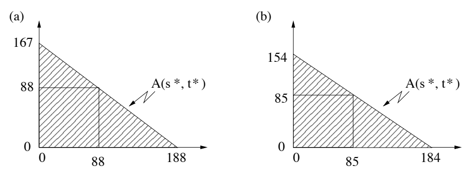
ECE 567 COMMUNICATION NETWORK ANALYSIS **FALL 2006**
SOLUTIONS TO PROBLEM SET 5

1. Effective bandwidth, bufferless link
(a) $\alpha_1(s) = \frac{1}{s}\ln(\frac{e^{2s}-1}{2s})$, $\alpha_2(s) = \frac{-\ln(1-s)}{s}$.
(b) n=79.05 (or round to $n = 79$)
(c) $\alpha_1(s^*) = 1.05$, $\alpha_2(s^*) = 1.19$, and C_{eff}=177.0. As one might expect, $\alpha_1(s^*)$ is somewhat smaller than $\alpha_2(s^*)$.



2. Effective bandwidth for a buffered link and long range dependent Gaussian traffic
(a) $n = 88.80$ (round down to n=88). $\alpha_1(s^*) = 1.0085$, $\alpha_2(s^*) = 1.1354$, $t^* = 31.6$, $C_{eff} = 190.38$.

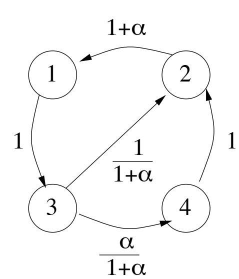


(b) $n = 85.33$ (round down to n=85). $\alpha_1(s^*) = 1.015$, $\alpha_2(s^*) = 1.1827$, $t^* = 22.7$, $C_{eff} = 187.53$. Due to the tighter overflow probability constraint, the effective bandwidths in part (b) are larger than those in part (a) and the effective bandwidth of the link is smaller. Interestingly, the critical time scale is smaller for part (b) than for part (a).

3. Time reversal of a simple continuous time Markov process

$$Q' = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3(1+\alpha)} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{\alpha}{3(1+\alpha)} \end{pmatrix}^{-1} Q^T \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3(1+\alpha)} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{\alpha}{3(1+\alpha)} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1+\alpha & -(1+\alpha) & 0 & 0 \\ 0 & \frac{1}{1+\alpha} & -1 & \frac{\alpha}{1+\alpha} \\ 0 & 1 & 0 & -1 \end{pmatrix}$$



4. Time reversibility of an M/GI/1 processor sharing queue
(a) No, due to the memory in the service time varibles. One Markov process is $\Gamma(t) = (N(t), S_1(t), \ldots, S_{N(t)}(t))$, where $S_1(t) \geq \cdots \geq S_{N(t)}(t)) \geq 0$ and $S_i(t)$ denotes how much service the ith customer in the system received so far.
(b) Let $\boldsymbol{n} = (n_1, \ldots, n_k)$ and $n = n_1 + \cdots + n_k$. The nonzero off-diagonal rates are

$$q(\boldsymbol{n}, \boldsymbol{n'}) = \begin{cases} \frac{n_i\mu_i}{n} & \text{if } \boldsymbol{n'} = (n_1, \ldots, n_i - 1, n_{i+1} + 1, \ldots, n_k) \\ \lambda & \text{if } \boldsymbol{n'} = (n_1 + 1, n_2, \ldots, n_k) \\ \frac{n_k\mu_k}{n} & \text{if } \boldsymbol{n'} = (n_1, \ldots, n_{k-1}, n_k - 1) \end{cases}$$

(c) The conjectured rate matrix for the time-reversed process is given by (only nonzero off-diagonal rates are indicated):

$$q'(\boldsymbol{n'}, \boldsymbol{n}) = \begin{cases} \frac{(1+n_{i+1})\mu_{i+1}}{n} & \text{if } \boldsymbol{n'} = (n_1, \ldots, n_i - 1, n_{i+1} + 1, \ldots, n_k) \\ \frac{(1+n_1)\mu_1}{n+1} & \text{if } \boldsymbol{n'} = (n_1 + 1, n_2, \ldots, n_k) \\ \lambda & \text{if } \boldsymbol{n'} = (n_1, \ldots, n_{k-1}, n_k - 1) \end{cases}$$

To simultaneously prove the conjecture for π and q', it suffices to check $\pi(\boldsymbol{n})q(\boldsymbol{n}, \boldsymbol{n'}) = \pi(\boldsymbol{n'})q'(\boldsymbol{n'}, \boldsymbol{n})$ for $\boldsymbol{n} \neq \boldsymbol{n'}$. We find

$$\begin{aligned} \frac{\pi(\boldsymbol{n'})}{\pi(\boldsymbol{n})} &= \begin{cases} \frac{a_{i+1}}{n_{i+1}+1}\frac{n_1}{a_i} = \frac{n_i\mu_i}{(n_{i+1}+1)\mu_{i+1}} & \text{if } \boldsymbol{n'} = (n_1, \ldots, n_i - 1, n_{i+1} + 1, \ldots, n_k) \\ \frac{(n+1)a_1\rho}{n_1+1} = \frac{(n+1)\lambda}{\mu_1(n_1+1)} & \text{if } \boldsymbol{n'} = (n_1 + 1, n_2, \ldots, n_k) \\ \frac{n_k}{\rho n a_k} = \frac{\mu_k n_k}{\lambda n} & \text{if } \boldsymbol{n'} = (n_1, \ldots, n_{k-1}, n_k - 1) \end{cases} \\ &= \frac{q(\boldsymbol{n}, \boldsymbol{n'})}{q'(\boldsymbol{n'}, \boldsymbol{n'})} \end{aligned}$$

as required.
(d) The total service time for a customer has the same distribution in either system, namely the sum $Exp(\mu_1) + \cdots + Exp(\mu_k)$. Hence, the random processes giving the number of customers in the system are statistically the same for the forward and reverse time systems.

5. A sequence of symmetric star shaped loss networks
(a) The set $\mathcal{R}$ of routes is the set of subsets of $\{1, \ldots, M\}$ of size two. Thus, $|\mathcal{R}| = \binom{M}{2}$. The process at time t has the form $X(t) = (X_r(t) : r \in \mathcal{R})$ with state space $\mathcal{S} = \{x \in \mathbb{Z}_+^{\mathcal{R}} : \sum_{r:i\in r} x_r \leq 5 \text{ for } 1 \leq i \leq M\}$ and transition rate matrix given by (only nonzero off diagonal rates are given):

$$\begin{aligned} q(x, x + e_r) &= \tfrac{4}{M-1} && \text{if } x, x + e_r \in \mathcal{S} \\ q(x, x - e_r) &= x_r && \text{if } x, x - e_r \in \mathcal{S} \end{aligned}$$

(b) The unthinned rate at a link ℓ is given by $\lambda_\ell = \sum_{r:\ell\in r} \nu_\ell = |\{r : r \in \ell\}| \times \frac{4}{M-1} = 4$, so for any M, $B_\ell \leq E[4, 5] = 0.19907$ and the call acceptance probability is greater than or equal to $(1 - 0.19907)^2 = 0.64149$. If $M = 2$ then $B_\ell = E[4, 5]$ and the call acceptance probability is 1-0.19907=0.8001.
(c) The reduced load approximation is given by the fixed point equations:

$$\widehat{\lambda_\ell} = \lambda(1 - \widehat{B_\ell}) \quad \text{and} \quad \widehat{B_\ell} = E[\widehat{\lambda_\ell}, C]$$

The equations do not depend on M, and the solution is $\widehat{B_\ell} = 0.1464$, leading to the estimate $(1 - \widehat{B_\ell})^2 = 0.72857$ for the call acceptance probability. The fixed point approximation for this example, for both $\widehat{B_\ell}$ and call acceptance probability, can be shown to be exact in the limit $M \to \infty$. The numbers are summarized in the following table:

	B_ℓ	$P\{\text{route } r \text{ is free}\}$
bound	≤ 0.1999	≥ 0.6915
$M = 2$	0.199	0.811
$M = 3$	0.14438	0.745
$M = 4$	0.146	0.739
$M \to \infty$	0.146	0.728

6. Deterministic delay constraints for two servers in series

(a) The total arrival stream to the first queue is $(\sigma_1 + \sigma_2, \rho_1 + \rho_2)$ constrained, and the server is FIFO, so the delay is bounded above by $d_1 = \lceil \frac{\sigma_1+\sigma_2}{C_1} \rceil$. ($d_1 = 2$ for the example.)

(b) Stream 1 at the output of queue 1 is $(\sigma_1 + \rho_1 d_1, \rho_1)$-upper constrained, because A_1 is (σ_1, ρ_1)-upper constrained and delay in the first queue is less than or equal to d_1. ($\sigma = \sigma_1 + \rho_1 d_1 = 8$ for the example.)

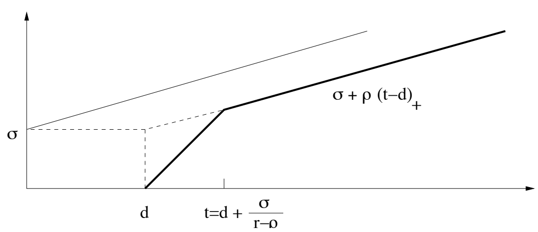
(c) By Example 2.3.13, p. 61 of Chang's book (or as discussed in class) there is a bound on the delay of the lower priority stream for a FIFO server, constant service rate C, and (σ, ρ) constrained inputs. This yields:

$$\text{delay for stream 1 in queue 2} \leq \frac{(\sigma_1 + \rho_1 d_1) + \sigma_3}{C_2 - \rho_3} = \frac{12}{3} = 4 \quad \text{(for the example)}$$

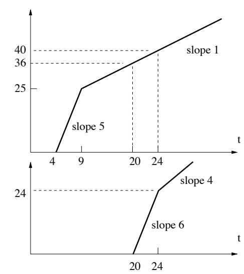
7. Calculation of an acceptance region based on the SCED algorithm

(a) Let $g(t) = \sigma + \rho t$ for $\rho, \sigma > 0$. By the definition given in the notes, $g^*(0) = 0$ and for $t \geq 1$,
$g^*(t) = \min\{g(s_1) + \cdots + g(s_n) : n \geq 1,\ s_1, \ldots, s_n \geq 1,\ s_1 + \cdots + s_n = t\} = \min\{\sigma n + \rho t : n \geq 1\} = \sigma + \rho t$.
Another approach to this problem is to use the definition of g^* in Chang's book. Yet another approach is to let $g_0(t) = I_{\{t \geq 1\}} g(t)$ and show that g_0 is the maximal subadditive function equal to 0 at 0, and less than or equal to g.

(b) Before working out special cases, let's work out the general case. Let $g(t) = \sigma + \rho t$ and $f(t) = (t - d)_+$ where $\sigma > 0, D > 0, r > \rho > 0$. To find $g^* \star f$ by a graphical method, we use the formula $(g^* \star f)(t) = \min_{0 \leq s \leq t}\{g^*(s) + f(t - s)\}$. That is, $g^* \star f$ is the minimum of the functions obtained by considering $g^*(s) + f(t - s)$ as a function of t on the interval $t \geq s$ for each s fixed.



So, for the parameters at hand,



(c) Since $n_1 g_1^* \star f_1 + n_2 g_2^* \star f_2$ is piecewise linear with intial value 0, this function is less than or equal to Ct for all $t \geq 0$ if inequality is true at breakpoints and at $+\infty$ (i.e. in the limit as $t \to \infty$). This requires $n_1(25, 36, 40, 1) + n_2(0, 0, 24, 4) \leq (900, 2000, 2400, 100)$, which simplifies to $n_1 \leq 36$ and $n_1 + 4n_2 \leq 100$.