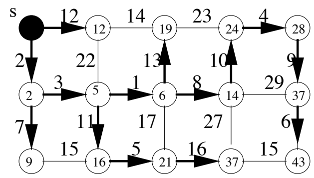
**ECE 467 COMMUNICATION NETWORK ANALYSIS** **FALL 2006**
**PROBLEM SET 6 SOLUTIONS**

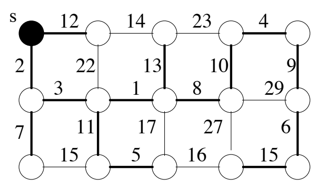
### 1. A shortest path problem

(a)




(b) Yes, because $D_j > D_i + d_{ij}$ for every edge $ij$ not in the tree.
(c) Synchronous Bellman-Ford would take 8 iterations (on the 9th iteration no improvement would be found) because the most hops in a minimum cost path is 8.

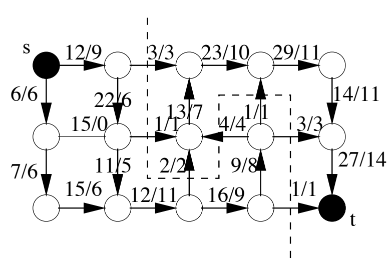
### 2. A minumum weight spanning tree problem




The MWST is unique because during execution of the Prim-Dijkstra algorithm there were no ties. (The sufficiency of this condition can be proved by a slight modification of the induction proof used to prove the correctness of the Prim-Dijkstra algorithm. Suppose the Prim-Dijkstra algorithm is executed. Prove by induction the following claim: The set of $k$ edges found after $k$ steps of the algorithm is a subset of *all* MWST's.)

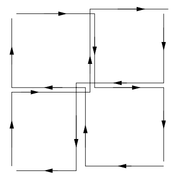
### 3. A maximum flow problem

The capacities, a maximum flow, and a minimum cut, are show. The maximum flow value is 15.

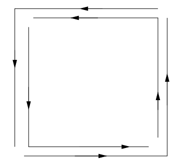



### 4. A flow deviation problem (Frank-Wolfe method)

(a) Initially the 8 links directed counter clockwise along the perimeter of the network carry zero flow, and the other 16 links carry flow $b$. So cost($f$)=$8b^2$.
(b) Flow $\overline{f}$ uses the 8 links not used initially, as shown.



f



$\overline{\text{f}}$

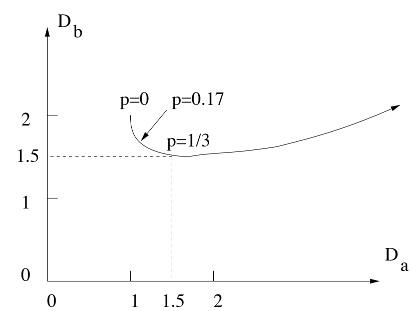
Thus, $\text{cost}(\alpha\overline{f} + (1-\alpha)f) = 8(2\alpha b)^2 + 16((1-\alpha)b)^2$. Minimizing over $\alpha$, we find $\alpha^* = \frac{1}{3}$. Under the new flow, $f_1 = \alpha^*\overline{f} + (1-\alpha^*)f$, all 24 links carry flow $\frac{2b}{3}$. The first derivative link length of all paths used is minimum. So $f_1$ is an optimal flow.

**5. A simple routing problem in a queueing network**

In equilibrium, the stations are independent and have the distribution of isolated M/M/1 queues, and the delay in an $M/M/1$ queue with arrival rate $\lambda$ and departure rate $\mu$ is $\frac{1}{\mu-\lambda}$. Therefore,

$$D_a = \frac{1}{2-(1+p)} \quad \text{and} \quad D_b(p) = p\left(\frac{1}{2-(1+p)}\right) + (1-p)\left(\frac{1}{2-(1-p)} + \frac{1}{2-(1-p)}\right)$$

(b)




(c) $p = 0$ minimizes $D_a$, because $D_a$ is increasing in $p$.
(d) Setting $D_b'(p) = 0$ yields $p = \frac{1}{3}$.
(e) Setting $D_a'(p) + D_b'(p) = 0$ yields $p \approx 0.17$. Only one iteration of the flow deviation algorithm is necessary because the only degree of freedom is the split for flow $b$ between two paths. The line search for flow deviation solves the problem in one iteration.
(f) The solution is $p = \frac{1}{3}$, because $D_a$ is increasing in $p$, $D_b$ is decreasing in $p$, and $D_a(\frac{1}{3}) = D_b(\frac{1}{3})$. (It is a coincidence that the answers to (d) and (f) are the same.)

**6. A joint routing and congestion control problem**

(a) Since $D_l'(F) = F$, the price for a link is equal to the flow on the link. For example, path 124 has price $2a + d$. The optimality conditions are:

$$\begin{aligned}
2a + d &\geq \frac{\beta_{14}}{a+b} \quad \text{with equality if } a > 0 \\
2b + c &\geq \frac{\beta_{14}}{a+b} \quad \text{with equality if } b > 0 \\
2c + b &\geq \frac{\beta_{23}}{c+d} \quad \text{with equality if } c > 0 \\
a + 2d &\geq \frac{\beta_{23}}{c+d} \quad \text{with equality if } d > 0
\end{aligned}$$

The first two conditions are for o-d pair $w = (1,2)$ and the last two are for o-d pari $w = (2,3)$.
(b) If $a, b, c, d > 0$ the equality constraints hold in all four conditions. The first two conditions yield $2a + d = 2b + c$ or, equivalently, $2a - 2b = c - d$. The last two yield $2c + b = a + 2d$ or, equivalently, $2a - 2b = 4c - 4d$. Thus, $c - d = 4c - 4d$ or, equivalently, $c = d$, and hence also $a = b$. (c) We seek a solution such that all four flows are nonzero, so that part (b) applies. Replacing $b$ by $a$ and $d$ by $c$, and using $\beta_{14} = 66$ and $\beta_{23} = 130$, the optimality conditions become

$$\begin{aligned} 2a + c &= \frac{66}{2a} \\ 2c + a &= \frac{130}{2c} \end{aligned}$$

or $a = b = 3$ and $c = d = 5$. To double check, note that the two routes for $w = (1,4)$ have price 11 each, which is also the marginal value for route $(1,4)$, and the two routes for $w = (2,3)$ have price 13 each, which is also the marginal value for route $(2,3)$.

**7. Sufficiency of the optimality condition for hard link constraints**
Since $L(x,p)$ is concave in $x$, for any $\epsilon$ with $0 < \epsilon < 1$,

$$L(x,p) - L(x^*,p) \leq \frac{1}{\epsilon}\left\{L(x^* + \epsilon(x - x^*), p) - L(x^*,p)\right\}$$

Taking the limit as $\epsilon \to 0$ yields

$$\begin{aligned} L(x,p) - L(x^*,p) &\leq \sum_r U_r'(x_r^*)(x_r - x_r^*) - \sum_l p_l(\sum_{r:l\in r} x_r - x_r^*) \\ &= \sum_r (x_r - x_r^*)\left\{U_r'(x_r^*) - \sum_{l\in r} p_l\right\} \leq 0 \end{aligned} \quad (1)$$

where the final inequality follows from the fact that for each $r$, the quantity in braces in (1) is less than or equal to zero, with equality if $x_r^* > 0$.
(b) Note that $\sum_l p_l(C_l - \sum_{r:l\in r} x_r^*) = 0$ and if $Ax \leq C$, then $\sum_l p_l(C_l - \sum_{r:l\in r} x_r) \geq 0$. So if $x \geq 0$ and $Ax \leq C$,

$$\sum_r U(x_r^*) = L(x^*,p) \geq L(x,p) \geq \sum_r U(x_r).$$

**8. Fair flow allocation with hard constrained links**
(a) By inspection, $x_{max-min} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
(b) (proportional fairness) Let $p_l$ denote the price for link $l$. Seek a solution to the equations

$$x_1 = \frac{1}{p_1 + p_2 + p_3} \quad x_2 = \frac{1}{p_1 + p_2} \quad x_3 = \frac{1}{p_1} \quad x_4 = \frac{1}{p_2 + p_3}$$

$$\begin{aligned} x_1 + x_2 + x_3 \leq 1, &\text{ with eqaulity if } p_1 > 0 \\ x_1 + x_2 + x_4 \leq 1, &\text{ with eqaulity if } p_2 > 0 \\ x_1 + x_4 \leq 1, &\text{ with eqaulity if } p_3 > 0 \end{aligned}$$

Clearly $x_1 + x_4 < 1$, so that $p_3 = 0$. Also, links 1 and 2 will be full, so that $x_3 = x_4$. But $x_3 = \frac{1}{p_1}$ and $x_4 = \frac{1}{p_3}$, so that $p_1 = p_2$. Finally, use $\frac{1}{2p_1} + \frac{1}{2p_1} + \frac{1}{p_1}$ to get $p_1 = p_2 = 2$, yielding $x_{pf} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$.
Flows 1 and 2 use paths with price $p_1 + p_2 = 4$ and each have rate $\frac{1}{4}$.
Flows 3 and 4 use paths with price $p_1 = p_2 = 2$ and each have rate $\frac{1}{2}$.