ANNALS OF MATHEMATICS
Vol. 54, No. 2, September, 1951

AN ITERATIVE METHOD OF SOLVING A GAME

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(July 28, 1950)

A two-person game[1] can be represented by its pay-off matrix $A = (a_{ij})$. The first player chooses one of the m rows and the second player simultaneously chooses one of the n columns. If the i^{th} row and the j^{th} column are chosen, then the second player pays the first player a_{ij}.

If the first player plays the i^{th} row with probability x_i and the second player plays the j^{th} column with probability y_j, where $x_i \geqq 0$, $\sum x_i = 1$, $y_j \geqq 0$, and $\sum y_j = 1$, then the expectation of the first player is $\sum\sum a_{ij}x_iy_j$. Furthermore,

$$\min_j \sum_i a_{ij} x_i \leqq \max_i \sum_j a_{ij} y_j, \tag{1}$$

since

$$\min_j \sum_i a_{ij} x_i \leqq \sum_i \sum_j a_{ij} x_i y_j \leqq \max_i \sum_j a_{ij} y_j.$$

The minimax theorem of game theory (see [1] page 153) asserts that for some set of probabilities $X = (x_1, \cdots, x_m)$ and $Y = (y_1, \cdots, y_n)$ the equality holds in (1). Such a pair (X, Y) is called a solution of the game. The value v of the game is defined by

$$v = \min_j \sum_i a_{ij} x_i = \max_i \sum_j a_{ij} y_j,$$

where (X, Y) is a solution of the game.

In this paper, we shall show the validity of an iterative procedure suggested by George W. Brown [2]. This method corresponds to each player choosing in turn the best pure strategy against the accumulated mixed strategy of his opponent up to then.

Let $A = (a_{ij})$ be an $m \times n$ matrix. $A_{i\cdot}$ will denote the i^{th} row of A and $A_{\cdot j}$, the j^{th} column. Similarly, if $V(t)$ is a vector, then $v_j(t)$ is the j^{th} component. Let $\max V(t) = \max_j v_j(t)$ and $\min V(t) = \min_j v_j(t)$. In this notation, (1) can be rewritten as follows:

$$\min \sum_i A_{i\cdot} x_i \leqq \max \sum_j A_{\cdot j} y_j, \tag{2}$$

whenever $x_i \geqq 0$, $\sum x_i = 1$, $y_j \geqq 0$, and $\sum y_j = 1$.

DEFINITION 1. A system (U, V) consisting of a sequence of n-dimensional vectors $U(0), U(1), \cdots$ and a sequence of m-dimensional vectors $V(0), V(1), \cdots$ is called a *vector system* for A provided that

$$\min U(0) = \max V(0),$$

[1] More technically, a finite two-person zero-sum game. See [1] in the bibliography at the end of the paper.

and

$$U(t+1) = U(t) + A_{i\cdot}, \qquad V(t+1) = V(t) + A_{\cdot j},$$

where i and j satisfy the conditions

$$v_i(t) = \max V(t), \qquad u_j(t) = \min U(t).$$

Thus a vector system for A can be formed recursively from a given $U(0)$ and $V(0)$. At each step, the row added to U is determined by a maximum component of V and the column added to V is determined by a minimum component of U.

An alternate notion of vector system is obtained if the condition on j in Definition 1 is replaced by

$$u_j(t+1) = \min U(t+1).$$

A vector system of this new type can also be built up recursively. The only difference is that here successive U and V are determined alternately while in the other definition U and V could be obtained simultaneously. In all the following proofs and theorems, either definition may be used.

In the special case $U(0) = 0$ and $V(0) = 0$, we see that $U(t)/t$ is a weighted average of the rows of A and $V(t)/t$ is a weighted average of the columns. Hence for every t and t',

$$\frac{\min U(t)}{t} \leqq v \leqq \frac{\max V(t')}{t'}.$$

If for some t and t', these two bounds are equal, we have a solution of the game. Unfortunately, this is not always the case. However George Brown [2] conjectured that as t and t' tend to ∞, the two bounds approach v. The main result of this paper is to prove this for any vector system. In numerical examples, vector systems of the second kind appear to converge more rapidly than the first.

THEOREM.[2] *If (U, V) is a vector system for A, then*

$$\lim_{t\to\infty} \frac{\min U(t)}{t} = \lim_{t\to\infty} \frac{\max V(t)}{t} = v.$$

The proof will be divided into four lemmas.

LEMMA 1. *If (U, V) is a vector system for a matrix A, then*

$$\liminf_{t\to\infty} \frac{\max V(t) - \min U(t)}{t} \geqq 0.$$

PROOF. For each t,

$$V(t) = V(0) + t\sum_j y_j A_{\cdot j} \text{ where } y_j \geqq 0, \sum y_j = 1,$$

[2] The solution to Problem 5 in the RAND Mathematical Problem Series II is contained as a special case of this theorem.

and

$$U(t) = U(0) + t\sum_i x_i A_{i\cdot} \text{ where } x_i \geqq 0, \sum x_i = 1.$$

Hence

$$\max V(t) \geqq \min V(0) + t \max \sum_j y_j A_{\cdot j} \geqq \min V(0) + tv,$$

and

$$\min U(t) \leqq \max U(0) + t \min \sum_i x_i A_{i\cdot} \leqq \max U(0) + tv.$$

Therefore,

$$\liminf_{t\to\infty} \frac{\max V(t) - \min U(t)}{t} \geqq 0.$$

Definition 2. If (U, V) is a vector system for A, then we say that the i^{th} row is eligible in the interval (t, t') provided that there exists t_1 with

$$t \leqq t_1 \leqq t'$$

and

$$v_i(t_1) = \max V(t_1).$$

Similarly, the j^{th} column is eligible in the interval (t, t') if there exists t_2 with

$$t \leqq t_2 \leqq t'$$

and

$$u_j(t_2) = \min U(t_2).$$

Lemma 2. *Given a vector system (U, V) for A, then if all the rows and columns of A are eligible in the interval $(s, s + t)$,*

$$\max U(s + t) - \min U(s + t) \leqq 2at$$

and

$$\max V(s + t) - \min V(s + t) \leqq 2at,$$

where

$$a = \max_{i,j} | a_{ij} | .$$

Proof. Let j be such that

$$u_j(s + t) = \max U(s + t).$$

Choose t' with $s \leqq t' \leqq s + t$ so that

$$u_j(t') = \min U(t').$$

Then

$$u_j(s + t) \leqq u_j(t') + at = \min U(t') + at,$$

since the change in the i^{th} component in t steps is not more than at. But

$$\min U(s + t) \geqq \min U(t') - at,$$

so that

$$\max U(s + t) - \min U(s + t) \leqq 2at.$$

Similarly,

$$\max V(s + t) - \min V(s + t) \leqq 2at.$$

LEMMA 3. *If all the rows and columns of A are eligible in $(s, s + t)$ for a given vector system (U, V), then*

$$\max V(s + t) - \min U(s + t) \leqq 4at.$$

PROOF. By Lemma 2,

$$\max V(s + t) - \min U(s + t) \leqq 4at + \min V(s + t) - \max U(s + t).$$

Hence it is sufficient to show that $\min V(s + t) \leqq \max U(s + t)$. Now applying (2) to the transpose of A, we have

$$\min \sum_j A_{\cdot j} y_j \leqq \max \sum_i A_{i \cdot} x_i,$$

whenever $x_i \geqq 0$, $\sum x_i = 1$, $y_j \geqq 0$, and $\sum y_j = 1$. In particular, choose x_i and y_j satisfying

$$U(s + t) = U(0) + (s + t) \sum A_{i \cdot} x_i,$$
$$V(s + t) = V(0) + (s + t) \sum A_{\cdot j} y_j.$$

Then

$$\begin{aligned} \min V(s + t) &\leqq \max V(0) + (s + t) \min \sum A_{\cdot j} y_j \\ &\leqq \min U(0) + (s + t) \max \sum A_{i \cdot} x_i \\ &\leqq \max U(s + t). \end{aligned}$$

LEMMA 4. *To every matrix A and $\varepsilon > 0$, there exists t_0 such that for any vector system (U, V),*

$$\max V(t) - \min U(t) < \varepsilon t \qquad \textit{for } t \geqq t_0 .$$

PROOF. The theorem holds for matrices of order 1 since $U(t) = V(t)$ for all t. Assume the theorem holds for all submatrices of A, then we will show by induction that it holds for A. Choose t^* so that for any vector system (U', V') corresponding to a submatrix A' of A, we have

$$\max V'(t) - \min U'(t) < \tfrac{1}{2}\varepsilon t \qquad \text{whenever } t \geqq t^*.$$

We shall prove that if in the given vector system (U, V) for A, some row or column is not eligible in the interval $(s, s + t^*)$, then

$$\max V(s + t^*) - \min U(s + t^*) < \max V(s) - \min U(s) + \tfrac{1}{2}\varepsilon t^*. \tag{3}$$

Suppose, for example, that the k^{th} row is not eligible in the interval $(s, s+t^*)$. Then we can construct a vector system (U', V') for the matrix A' obtained by deleting the k^{th} row of A, in the following way:

$$U'(t) = U(s+t) + C,$$
$$V'(t) = \text{Proj}_k V(s+t) \qquad \text{for } t = 0, 1, \cdots, t^*,$$

where C is the n-dimensional vector all of whose components are equal to $\max V(s) - \min U(s)$ and $\text{Proj}_k V$ is the vector obtained from V by omitting the k^{th} component. The rows of A' will be numbered $1, 2, \cdots, k-1, k+1, \cdots, m$. Notice first that $\min U'(0) = \max V'(0)$. Furthermore, if

$$U(s+t+1) = U(s+t) + A_{i\cdot}, \qquad V(s+t+1) = V(s+t) + A_{\cdot j},$$

then

$$U'(t+1) = U'(t) + A'_{i\cdot}, \qquad V'(t+1) = V'(t) + A'_{\cdot j}.$$

Also $v_i(s+t) = \max V(s+t)$ if and only if $v'_i(t) = \max V'(t)$ and $u_j(s+t) = \min U(s+t)$ if and only if $u'_j(t) = \min U'(t)$ for $0 \leqq t \leqq t^*$. Hence we see that U' and V' must satisfy the recursive restrictions of the definition of a vector system for $0 \leqq t \leqq t^*$, since U and V do. Naturally, we may continue U' and V' indefinitely to form a vector system for A'.

Now by the choice of t^*, we know that

$$\max V'(t^*) - \min U'(t^*) < \tfrac{1}{2}\varepsilon t^*.$$

Hence

$$\begin{aligned}\max V(s+t^*) &- \min U(s+t^*) \\ &= \max V'(t^*) - \min U'(t^*) + \max V(s) - \min U(s) \\ &< \max V(s) - \min U(s) + \tfrac{1}{2}\varepsilon t^*.\end{aligned}$$

We can now show that given any vector system (U, V) for A,

$$\max V(t) - \min U(t) < \varepsilon t \qquad \text{for } t \geqq 8at^*/\varepsilon.$$

Consider $t > t^*$. Let θ with $0 \leqq \theta < 1$ and q a positive integer be so chosen that $t = (\theta + q)t^*$.

CASE 1. Suppose there is a positive integer $s \leqq q$ so that all rows and columns of A are eligible in the interval $((\theta + s - 1)t^*, (\theta + s)t^*)$. Take the largest such s, then

$$\begin{aligned}(4)\quad &\max V(t) - \min U(t) \\ &\leqq \max V((\theta+s)t^*) - \min U((\theta+s)t^*) + \tfrac{1}{2}\varepsilon(q-s)t^*.\end{aligned}$$

We obtain this inequality by repeated application of (3), since in each of the intervals

$$((\theta + r - 1)t^*, (\theta + r)t^*) \qquad \text{for } r = s+1, \cdots, q,$$

some row or column of A is not eligible. From Lemma 3 and the choice of s, we have

$$\max V((\theta + s)t^*) - \min U((\theta + s)t^*) \leqq 4at^* . \tag{5}$$

From (4) and (5), we obtain

$$\max V(t) - \min U(t) \leqq 4at^* + \tfrac{1}{2}\varepsilon(q - s)t^* < (4a + \tfrac{1}{2}\varepsilon q)t^* .$$

CASE 2. If there is no such s, then in each interval $((\theta + r - 1)t^* , (\theta + r)t^*)$ some row or column of A is not eligible. Hence

$$\max V(t) - \min U(t) < \max V(\theta t^*) - \min U(\theta t^*) + \tfrac{1}{2}\varepsilon q t^* \leqq 2a\theta t^* + \tfrac{1}{2}\varepsilon q t^*.$$

Therefore, in either case,

$$\max V(t) - \min U(t) < (4a + \tfrac{1}{2}\varepsilon q)t^* \leqq 4at^* + \tfrac{1}{2}\varepsilon t < \varepsilon t \qquad \text{for } t \geqq 8at^*/\varepsilon.$$

From Lemmas 1 and 4, we see that

$$\lim_{t\to\infty} \frac{\max V(t) - \min U(t)}{t} = 0.$$

But from (1),

$$\limsup_{t\to\infty} \frac{\min U(t)}{t} \leqq v,$$

$$\liminf_{t\to\infty} \frac{\max V(t)}{t} \geqq v.$$

Hence

$$\lim_{t\to\infty} \frac{\min V(t)}{t} = \lim_{t\to\infty} \frac{\max V(t)}{t} = v,$$

which completes the proof of the theorem.

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