MULTISTAGE GAME MODELS AND DELAY SUPERGAMES

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ABSTRACT

The order of stages in a multistage game is often interpreted by looking at earlier stages as involving more long term decisions. For the purpose of making this interpretation precise, the notion of a delay supergame of a bounded multistage game is introduced. A multistage game is bounded if the length of play has an upper bound. A delay supergame is played over many periods. Decisions on all stages are made simultaneously, but with different delays until they become effective. The earlier the stage the longer the delay.

A subgame perfect equilibrium of a bounded multistage game generates a subgame perfect equilibrium in every one of its delay supergames. This is the first main conclusion of the paper. A subgame perfect equilibrium set is a set of subgame perfect equilibria all of which yield the same payoffs, not only in the game as a whole, but also in each of its subgames. The second main conclusion concerns multistage games with a unique subgame perfect equilibrium set and their delay supergames which are bounded in the sense that the number of periods is finite. If a bounded multistage game has a unique subgame perfect equilibrium set, then the same is true for every one of its bounded delay supergames.

Finally the descriptive relevance of multistage game models and their subgame perfect equilibria is discussed in the light of the results obtained.

1. Introduction

In the economic literature one finds many game models, in which the players make simultaneous decisions on each of a number of successive stages, always fully informed about what has been done on previous stages. An early example can be found in my paper " a simple model of imperfect competition where 4 are few and six are many (Selten, 1973). In this oligopoly model the players first decide whether they want to take part in cartel bargaining. This participation stage is followed by a cartel bargaining in which quota cartels can be proposed and agreed upon. The third and last stage is a supply decision stage in which production quantities are fixed.

Another example is a two stage duopoly model with production capacity decisions on the first stage and Bertrand price competition on the second stage (Kreps and Scheinkman, 1983). Multistage game models can be analyzed on the basis of the subgame perfect equilibrium concept (Selten, 1965), the simplest refinement of ordinary game theoretic equilibrium (Nash, 1951). Sometimes additional selection criteria are combined with subgame perfect equilibria, like symmetry and local efficiency in the case of my above mentioned model. The analysis of my model yields the result that up to 4 competitors always form a cartel whereas in the presence of 6 or more competitors the probability of cartel formation is less than 2%. The model by Kreps and Scheinkman deepens our understanding of Cournot's (1838) oligopoly theory. In equilibrium Cournot quantity competition takes place on the first stage of their model. The examples show that the analysis of multistage game models can lead to interesting theoretical conclusions.

In some cases it may be justified to look at the stages of a multistage game model as decision points succeeding each other in time. However, often this direct temporal interpretation is not adequate. Many multistage game models do not really aim at the description of situations in which decisions must be made in a fixed time order. Thus the model of Kreps and Scheinkman places capacity choice before price choice, not because price choice cannot precede capacity choice, but rather because capacity decisions are considered to be more long term than price decisions. It seems to be natural to consider long term decisions as fixed, when short term decisions are made. "Term length" in the sense of a position on a short term-long term scale rather than time is the consideration underlying the order of stages. If decisions of greater term length are modelled as made on an earlier stage, this is intended to have the effect that in subgame perfect equilibrium more short term decisions are based on more long term decisions taken as fixed.

The term length interpretation looks at a multistage game model as a condensed description of an ongoing situation in which stage decisions are not made once and for all. Strategic variables may change over time, but more short term decisions are in some way subordinated to more long term ones. The reasons for this subordination are not explicitly spelled out.

Simple oligopoly models like the Cournot model which involve only one stage are usually also interpreted as condensed descriptions of ongoing situations. The literal interpretation as a one-shot game would leave little room for applied significance. One must think of such models as being played repeatedly in a supergame (Aumann, 1959, Friedman, 1977), but without making use of the potential for quasicooperation which may be present in such situations. The analysis of the one shot game instead of the supergame amounts to the assumption that any kind of collusion is excluded by effectively enforced cartel laws or for other reasons.

Similarly one could look at a multistage game model as representing the structure of a period in a supergame. However, this would mean that one sticks to the direct temporal interpretation of the order of stages. Obviously the term length interpretation requires a different picture of the underlying ongoing situation. It is the modest purpose of this lecture to provide an explicit model of the underlying situation which justifies the reduction to the multistage game with its subordination of the shorter term to the longer term.

What does it mean in the model by Kreps and Scheinkman that capacity decisions are long term and price decisions are short term? It could mean that due to exogenous institutional circumstances capacities can be adjusted only at certain points in time, say at the beginning of the year, whereas prices can be changed every day. Alternatively one could think of a difference of change costs, high for capacity adjustments and low for price changes. This would lead to a model in the spirit of the inertia supergame (Marschak and Selten, 1978). A third possibility is the answer given here. It focusses on the delay between the time a decision is made and the time at which it becomes effective. This delay may be two years for capacity adjustments and only one day for price adjustments. The longer the delay the more long term the decision is considered to be.

It seems to me that in many cases the difference between a more long term and a more short term decision is adequately explained as a difference of delay. Of course, in some multistage game models necessities of temporal order, exogenous institutional circumstances and differences of change costs may also be considerations in the interpretation of the order of stages. However, here we shall only be concerned with the term length interpretation elaborated by looking at differences of term length as differences of delay times needed until a decision becomes effective. The underlying ongoing situation will be modelled as a special kind of game, called a "delay supergame". In a delay supergame decisions on all strategic variables are made at the same time, period after period, but these decisions become effective with different delays. Thus, in period t decisions on the price in t+l and capacity in t+10 may be made, on both variables at the same time and simultaneously by all players.

In a delay supergame the players have full information about previous history of play, but not about simultaneous decisions made by other players. All decisions made in a period become publicly known at the beginning of the next period.

It does not really matter exactly how long the delays are. For the analysis of delay supergames only the order of the decision variables with respect to delay length matters.

A delay supergame is not necessarily played for a fixed number of periods; the definition will involve a probability distribution over the number of periods played. We speak of a "bounded" delay supergame if the number of periods has a finite upper bound and of an "unbounded" one otherwise. The distinction between bounded and unbounded delay supergames is game theoretically important.

Every subgame perfect equilibrium of a bounded multistage game always generates a subgame perfect equilibrium for every one of its bounded or unbounded delay supergames. This is the first main conclusion of the paper (theorem 1 in 5.3). The generated equilibrium can roughly be described as the repeated application of the multistage game equilibrium strategies in every period played.

In general, a delay supergame and especially an unbounded one may have many additional subgame perfect equilibria. It is well known that this happens already in ordinary supergames (Rubinstein, 1976, 1980, Benoit and Krishna, 1985). Since normal form games are special multistage games with only one stage, supergames are special delay supergames.

A subgame perfect equilibrium set is a set of subgame perfect equilibria all of which yield the same payoffs not only in the game as a whole but also in each of its subgames. A multistage game or a delay supergame will be called "determinate" if the set of all of its subgame perfect equilibria is a subgame perfect equilibrium set. Every bounded delay supergame of a determinate bounded multistage game is determinate. This is the second main conclusion of this paper (theorem 2 in 5.5).

This lecture will not be concerned with the question which kind of "folk theorems" hold for which class of delay supergames. Such theorems are interesting from the point of view of normative game theory, but their applied significance is limited. Finite supergames of prisoners' dilemma games have only one subgame perfect equilibrium which prescribes the non-cooperative choice everywhere, but nevertheless experienced experimental subjects cooperate in such games until shortly before the end (Selten and Stoecker, 1986). On the other hand in some supergame-like oligopoly experiments cooperation is not observed (Sauermann and Selten, 1959, Hoggatt, 1959, Fouraker and Siegel, 1963, Stern, 1967). It is an empirical question under which conditions behavior in a delay supergame converges to a subgame perfect equilibrium of the underlying multistage game. At the end of the paper this problem will be discussed in more detail.

Instead of the usual framework of the extensive game (von Neumann and Morgenstern, 1944, Kuhn, 1953, Selten, 1975) a somewhat different one is used here, which is especially adapted to multistage games and their delay supergames. Simultaneous decisions are represented as being made at the same history of previous play and information is not explicitly modelled. A "choice set function" defined recursively together with a "path set" take over the role of the game tree. As in the usual extensive form a "probability assignment" describes the probability of random choices and a payoff function determines the payoffs at the end of a play. The framework could be made as general as that of an extensive game by the additional introduction of information partitions for the players but this will not be done here.

Even if our main conclusions are intuitively plausible and not surprising some formalism is necessary to make statements and their proofs precise.

2. Multistage games

A multistage game will be defined as a structure built up of four constituents, a start s, a choice set function A, a probability assignment p, and a payoff function h. The start s is a symbol which represents the situation before the beginning of the game. The choice set function describes what choices are available to the players in every situation which may arise in the game. The probability assignment p assigns probabilities to random choices and the payoff function h specifies the payoffs at the end of the game. Detailed formal definitions are given below.

2.1 The choice set function

A multistage game involves n personal players 1,...,n and a *random player 0* (interpreted as a random mechanism). In the following we present a joint recursive definition of a *choice set function* and the notion of a *path* (a path represents a previous history of play). In addition to this, further auxiliary definitions like that of a play, a preplay, and a choice combination are introduced. Interpretations are added in brackets.

1. The *start* s is a *path.*
2. If u is a path then the *choice set function* A assigns a *choice set* $A_i(u)$ to every player i = 0,...,n.

Auxiliary definitions and notations: A player i is called *active* at a path u if $A_i(u)$ is non-empty and *passive* otherwise (i = 0,...,n). A path u is a *play* if all players i = 0,...,n are passive at u and a *preplay* otherwise. (A play represents a history from the beginning to the end; a preplay still has to be continued.) The set of all active players at a path u is denoted by N(u). A *choice combination* at a preplay u is a system

$$a = (a_i)_{i \in N(u)} \text{ with } a_i \in A_i(u) \text{ for all } i \in N(u)$$

The set of all choice combinations at u is denoted by A(u).

3. If u is a preplay and a is a choice combination at u, then v = ua is a *path.* All paths are generated in this way.

Notation: The set of all paths is denoted by U, the set of all plays by Z and the set of all preplays by P . According to 3. a path u = s $a^1 \ldots a^k$ is built up as a sequence beginning with the start s and continued by successive choice combinations $a^1,\ldots,a^k$.

Finiteness of random choice sets: We only consider choice set functions with the additional property that all random choice sets $A_0(u)$ are finite. In this way we avoid tedious technicalities. In the following finiteness of all random choice sets will always be assumed.

2.2. The probability assignment

A probability assignment p assigns a probability distribution p_u, over $A_0(u)$ to every preplay at which the random player is active.

Auxiliary definitions and notation: The probability assigned to a choice $a_0 \in A_0(u)$ by p_u is denoted $p_u(a_0)$. The *length* of a path $u = sa^1...a^k$ is the number k of choice combinations following s in u. The length of u is denoted by $|u|$.

Comment: Our framework does not exclude multistage games without an upper bound on the length of a preplay. Multistage game models usually have a finite number of stages. Arbitrarily long preplays cannot arise in such models. However, we aim at a definition which also covers delay supergames without any bound on the number of periods. Unbounded delay supergames will involve stopping probabilities which have the effect that with probability 1 the game ends in finite time and that expected payoffs can be defined. In order to achieve this purpose for our general framework we impose a joint condition on the choice set function A and the probability assignment p.

Random stopping condition: A positive integer μ and a real number δ with $0 < \delta < 1$ exist such that for every preplay u with $|u| \geq \mu$ the choice set $A_0(u)$ contains a choice ω) with $p_u(\omega) \geq \delta$ and with the following property: If ω is the random player's component of $a \in A(u)$, then v = ua is a play.

We shall only consider multistage games, for which the random stopping condition is satisfied. It will always be assumed that this is the case.

2.3 The payoff function

A *payoff function* h is a function which assigns a *payoff vector*

$$h(z) = (h_1(z),..., h_n(z))$$

to every play $z \in Z$. The components $h_i(z)$ of h(z) are real numbers. $h_i(z)$ is *player i's payoff* for z.

Boundedness of payoffs: We only consider payoff functions with the property that constants C_0 and C_1 exist such that

$$|h_i(z)| \leq C_0 + C_1 |z|$$

holds for every play $z \in Z$ and for $i = 1,...,n$.

We impose this boundedness condition in order to make sure that expected payoffs can be defined. In view of the intended application to delay supergames, it is important to permit an increasing linear dependence on the length of a play. It will always be assumed that payoffs are bounded in this way.

2.4 Definition of a multistage game

A *multistage game*

$G = (s, A, p, h)$

is composed of four constituents, a start s, a choice set function A, a probability assignment p and a payoff function h, with the properties explained above (see 2.1, 2.2, and 2.3).

A multistage game G = (s, A, p, h) is called *bounded,* if in G the length of a path is bounded from above. Obviously in such games a maximum length M of a path exists. It can be seen immediately that the existence of this maximum length M implies that the random stopping condition of 2.2 holds with $\mu = M$ simply because there are no preplays u with $|u| \geq M$.

2.5 Strategies

In sections 2.5 - 2.7 all definitions will refer to a fixed but arbitrary multistage game G = (s, A, p, h). The set of all preplays at which player i is active is denoted by P_i. We call P_i *player i's preplay set.* A *local strategy* of a personal player i at a preplay $u \in P_i$ is a probability distribution b_{iu} over player i's choice set $A_i(u)$ which assigns positive probabilities to finitely many choices only. The probability assigned to a choice $a_i \in A_i(u)$ by b_{iu} is denoted by $b_{iu}(a_i)$.

A *behavior strategy* b_i of player i is a system of local strategies

$$b_i = (b_{iu})_{u \in P_i}$$

specifying a local strategy b_{iu} for every preplay u of player i. Player i's preplay set P_i may be empty. In this case the definition of a behavior strategy is to be understood in such a way that player i has exactly one behavior strategy, the *empty* strategy.

Comment: In multistage games every player is always fully informed about all choices on previous stages. This implies that such games have perfect recall. It is clear that the extensive games representing a multistage game have the formal property of perfect recall as it is usually expressed (Kuhn 1953, Selten 1975). Kuhn (1953) has proved a theorem which shows that without any essential loss the noncooperative analysis of finite extensive games with perfect recall can be restricted to behavior strategies. Aumann (1964) has generalized this theorem to extensive games in which a continuum of choices may be available in some choice sets. This is important in the context of multistage game models which usually involve continuously varying decision parameters.

In order to avoid tedious technical detail we shall restrict our attention to finite local strategies, i.e. local strategies with a finite carrier and finite behavior strategies which specify such local strategies only.

Further definitions: A local strategy is called *pure* if it assigns probability 1 to one of the choices. Pure local strategies can be identified with choices. The set of all finite local strategies of a player i at a preplay $u \in P_i$ is denoted by B_{iu}. The set of all behavior strategies of a personal player i is denoted by B_i. A *pure strategy* of a personal player i assigns a pure local strategy or, in other words, a choice at u to every $u \in P_i$.

A *strategy combination* $b = (b_1,...,b_n)$ is an n-tuple specifying a behavior strategy b_i for every personal player. A strategy combination is called *pure,* if all its components are pure. The set of all strategy combinations is denoted by B.

2.6 Realization probabilities

Consider a strategy combination $b = (b_1,..., b_n)$ and a preplay u. For every personal player i active at u let b_{iu} be the local strategy assigned to u by b_i. For every choice combination

$$a = (a_i)_{i \in N(u)} \text{ with } a_i \in A_i(u)$$

we define the *conditional realization probability* of a at u as the product of $p_0(a_0)$ and all $b_{iu}(a_i)$ with i active at u. This probability is denoted by $b_u(a)$:

$$b_u(a) = p_u(a_0) \prod_{i \in N(u)-\{0\}} b_{iu}(a_i)$$

In this way a probability distribution b_u over A(u) is associated to a strategy combination $b = (b_1,..., b_n)$ and a preplay u.

Now consider a path $v \in V$ with $v = sa^1...a^k$. We say that a path $u = sa^1...a^j$ is *on* v if we have $j \leq k$ and the first choice combinations $a^1,...,a^j$ are the same in u and v. The *realization probability of v under* $b = (b_1,...,b_n)$ is the product of all $b_u(a)$ with ua on v: This probability is denoted by b(v).

$$b(v) = \prod_{ua \text{ on } v} b_u(a)$$

The realization probability of $v = sa^1...a^k$ is interpreted as the probability that in the course of playing the game with the strategies in b the play passes the choice combinations $a^1,...,a^k$ one after the other.

2.7 Expected payoffs

For every strategy combination $b = (b_1,...,b_n)$ we shall define *expected payoffs* $H_i(b)$ for every personal player i = 1,...,n. We shall focus on a fixed but arbitrary personal player i. In the case of a bounded multistage game G player i's expected payoffs are defined as follows:

$$H_i(b) = \sum_{z \in Z} b(z) h(z)$$

In the following we shall assume that G is an unbounded multistage game. In this case the definition of expected payoffs is essentially the same as in the bounded case, but it needs to be elaborated, since infinite sums do not necessarily converge. For k = 0,1,... let Z_k, be the set of all plays z with $|z| = k$. Player i's expected payoff $H_i(b)$ is defined as

$$H_i(b) = \lim_{T\to\infty} \sum_{k=0}^{T} \sum_{z\in Z} b(z)h(z)$$

It has to be shown that the limit exists. For k = 0, 1,... let Q_k be the set of all preplays v with $|v| = k$. Define

$$b(Z_k) = \sum_{z\in Z_k} b(z)$$

$$b(Q_k) = \sum_{v\in Q_k} b(v)$$

$$Y_k = \sum_{z\in Z_k} b(z)h_i(z)$$

Since all local strategies assign positive probabilities to finitely many choices only, these sums are finite. Let μ and δ be numbers such that the random stopping condition holds with these numbers. The random stopping condition permits the conclusion that for $k = \mu, \mu+1,...$ we have

$$b(Q_{k+1}) \leq (1-\delta)b(Q_k)$$

and therefore

$$b(Q_k) \leq (1-\delta)^{\kappa-\mu}b(Q_\mu)$$

Another conclusion from the second last inequality is the following one

$$b(Q_k) - b(Q_{k+1}) \leq \delta\, b(Q_k)$$

for $k = \mu, \mu+1,...$ On the left hand side of the last inequality we find nothing else than $b(Z_{k+1})$ since after the next choice combination a preplay in Q_k becomes a preplay in Q_{k+1} or a play in Z_{k+1}. In view of $b(Q_\mu) \leq 1$ the last two inequalities permit the following conclusion:

$$b(Z_{k+1}) \leq \delta\, (1-\delta)^{k-\mu}$$

for $k = \mu, \mu+1,...$. The boundedness condition for payoffs can now be used to bound Y_k:

$$|Y_k| \leq \delta(1-\delta)^{k-\mu-1}(C_0+C_1k) \quad \text{for } k = \mu, \mu+1, \mu+2,...$$

where C_0 and C_1 are numbers with the properties required by the boundedness condition on payoffs. It can be seen without difficulty that the infinite sum of the terms on the right-hand side converges. Therefore the same is true for the terms on the left-hand side. This has the consequence that the limit exists, by which $H_i(b)$ is defined. We call $H(b) = (H_1(b),...,H_n(b))$ the *expected payoff vector* of b. The function H which assigns H(b) to every $b \in B$ is referred to as the *expected payoff function.*

3. Equilibria

In this section we shall first define equilibrium in the framework of the multistage game. Then we look at subgames and subgame perfectness will be defined. As before, all definitions refer to a fixed but arbitrary multistage game G = (s, A, p, h).

3.1 Equilibrium

An *i-incomplete strategy combination* b_{-i} is an (n-1)-tupel of behavior strategies b_j with one strategy for all personal players except player i:

$$b_{-i} = (b_{1},..., b_{i-1}, -, b_{i+1},..., b_n)$$

We use the notation $b_i b_{-i}$ for the strategy combination b which contains b_i and the components b_j of b_{-i}. We say that a behavior strategy $\tilde{b}_i$ is a *best reply* to b_{-i}, if we have

$$H_i(\tilde{b}_i b_{-i}) = \max_{b_i \in B_i} H_i(b_i b_{-i})$$

$\tilde{b}_i$ is a *best reply* to $b = (b_1,...,b_n)$, if it is a best reply to the i-incomplete strategy combination b_{-i} formed by the components of b except b_i. We say that $\tilde{b} = (\tilde{b}_1,...,\tilde{b}_n)$ is a *best reply* to $b = (b_1,... b_n)$, if for i = 1,...,n the behavior strategy $\tilde{b}_i$ is a best reply to b.

A strategy combination b = (b,,... b,) is an *equilibrium,* if it is a best reply to itself. *An equilibrium set* E is a set of equilibria with the property that the payoff vector H(b) is the same one for all $b \in B$.

3.2 Subgames

For every preplay u we define a *subgame* G^u at u of G. This subgame is the multistage game

$$G^u = (u, A^u, p^u, h^u)$$

whose constituents will be described in the following. The start is the preplay u and A^u is the restriction of A to the paths v such that u is on v. A path v of this kind permits a representation of the form:

$$v = ua^1...a^k$$

The set of all these paths is denoted by U^u. Paths in U^u are at the same time paths in G starting with s and paths in G^u starting with u. The probability assignment p^u is the restriction of p^u to U^u. The set of all plays in G^u is denoted by Z^u. The payoff function h^u is the restriction of h to Z^u. It can be seen immediately that u, A^u, p^u and h^u form a multistage game with all the properties required in 2.1, 2.2, and 2.3.

Comment: Even if we did not formally describe how a multistage game is mapped to an equivalent extensive game, it can be seen without difficulties that the subgames defined above correspond to the subgames of an equivalent extensive form.

3.3 *Subgame perfectness*

We continue to look at a subgame G^u of G. The set of all preplays in U^U at which player i is active is denoted by P_i^u. The restriction of a behavior strategy b_i for G to P_i^u is a behavior strategy b_i^u for the subgame G^u. We say that b_i^u is the strategy *induced* by b_i on G^u. Similarly a strategy combination $b^u = (b_1^u,..., b_n^u)$ is *induced* by $b = (b_1,...,b_n)$ if for i = 1,..., n the behavior strategy b_i^u is induced by b_i on G^u. An i-incomplete strategy combination b^u_{-i} is *induced* by an i-incomplete strategy combination b_{-i} if every component of b^u_{-1} is induced by the corresponding component of b_{-1}. A set E^u of strategy combinations for G^u is *induced* by a set E of strategy combinations for G , if E^u is the set of all strategy combinations b^u induced by strategy combinations $b \in E$.

Let r_i be a best reply to a strategy combination $b = (b_1,...,b_n)$ of G. For every subgame G^u of G let r_i^u, be the strategy induced by r_i on G^u and $b^u = (b_1^u,..., b_n^u)$ the strategy combination induced by b on G^u. We say that r_i is a *subgame perfect best reply* to b if for every subgame G^u of G the behavior strategy r_i^u, is a best reply to b^u.

An equilibrium $\tilde{b} = (\tilde{b}_1,..., \tilde{b}_n)$ is *subgame perfect* if it induces an equilibrium on every subgame G^u of G or, in other words, if for i = 1,...,n the behavior strategy $\tilde{b}_i$ is a subgame perfect best reply to $\tilde{b}$. Similarly an equilibrium set E of *G* is *subgame perfect* if it induces an equilibrium set E^U on every subgame G^u of G.

4. *Delay supergames*

In this section delay supergames will be formally defined. A game of this kind is associated with a bounded multistage game G. In addition to the underlying game G the description of a delay supergame involves further specifications. It is necessary to specify delays after which decisions made on different stages become effective, stopping probabilities as a function of time and initial conditions on what is carried over from the past.

The interpretation of a delay supergame will focus on the case that in the underlying bounded multistage game K decision variables are fixed one after the other by some or all players in K successive stages. The definition of a multistage game does not exclude the possibility that different kinds of decisions have to be made at different preplays of the same length. However, in such cases it may not be adequate to assume that the delay after which a decision becomes effective depends only on the length of the preplay at which it is made.

In section 4 all definitions refer to a fixed but arbitrary bounded multistage game G = (s, A, p, h) in which the maximum length of a play is K. A delay supergame of G is itself also a multistage game G* = (s*, A*, p*, h*) whose constituents are derived in a systematic manner from G and some additional specifications to be explained in 4.1 - 4.4.

4.1 Delay vector

In order to describe a delay supergame of G, it is necessary to specify a *delay vector*

$$m = (m_1, ..., m_k)$$

whose components are non-negative integers with $m_j > m_k$ for $j < k$:

$$m_1 > m_2 > ... > m_K \geq 0$$

The number m_k is called the *delay of stage* k. It is interpreted as the number of periods after which a decision on stage k becomes effective.

4.2 Stopping rule

A delay supergame of G begins with an *initial period* t_o and is played either for at most finitely many periods $t_o, ..., T$ or for potentially infinitely many periods $t_o, t_o + 1, ...$ In the first case the delay supergame is *bounded* and in the second case it is *unbounded.* In the bounded case a *reachable* period is one of the periods $t_o, ..., T$. In the unbounded case all periods $t_o, t_o + 1, ...$ are reachable. The set of all *reachable* periods is denoted by R.

A *stopping rule* w assigns a *stopping probability* w_t with

$$0 \leq w_t < 1$$

to every reachable period $t \in R$. The stopping probability w_t is interpreted as the conditional probability with which the delay supergame stops in period t, if period t-l has been reached. As will be explained later, no payoffs are obtained for period t if the game stops in period t. The players do not know whether the game stops in t when they make their choices at t. This depends on a random choice at t. All choices at t including this random choice are thought of as being made simultaneously. In the bounded case the game always stops after period T, if it is reached.

We think of w as a function together with the set R on which it is defined.

In the bounded case, T will always denote the *last reachable period.* In the unbounded case we shall always assume that w satisfies the following stopping requirement.

Stopping requirement: A real number δ with $0 < \delta < 1$ and a positive integer μ exist, such that the following is true:

$$w_t \geq \delta \quad \text{for all periods} \quad t = t_o + \mu,\ t_o + \mu + 1, \ldots$$

As we shall see later, this stopping requirement secures the random stopping condition of 2.2 for unbounded delay supergames.

4.3 The initial status assignment

In the following m will always be a fixed but arbitrary delay vector, t_o will stand for the initial period, and w will be a fixed but arbitrary stopping rule.

As in 2.7 the set of all preplays u of G with $|u| = k$ is denoted by Q_k and Z_k stands for the set of plays z of G with $|z| = k$. For k = 0,...K a *k-prearrangement* is either a preplay of G of length k or a play z of G with $|z| \leq k$. The set of all k-prearrangements is denoted by S_k. We say that a choice at a preplay u of G is made on *stage* k or that it is a *stage* k *decision* if u belongs to Q_{k-1}.

Consider a delay m_k and a reachable period $t_o+\tau$ with $t < m_k$ The interpretation of m_k as the delay until a stage-k-decision becomes effective suggests that a stage k decision for period $t_o+\tau$ should not be modelled as being made within the delay supergame but rather as predetermined by the past. Accordingly the definition of a delay supergame associated with G requires the specification of all decisions of this kind in a way which will be explained below.

For $\tau = 0,1,\ldots$ we define a *predetermination span* $\delta(\tau)$ interpreted as the highest stage k for which decisions for a period $t_o+\tau$ are excluded by $\tau < m_k$. For $\tau = 0,\ldots, m_1-1$ the predetermination span $\delta(\tau)$ is the index k of the smallest delay m_k with $\tau < m_k$. For $\tau = m_1, m_1+1,\ldots$ we define $\delta(\tau) = 0$.

An *initial status assignment* assigns an *initial status*

$$x(t,s^*) \in S_\tau \text{ with } \tau = \delta(t-t_o)$$

to every reachable period t. The initial status $x(t,s^*)$ is interpreted as the description of what is predetermined at period t. What is predetermined must be a $\delta(t-t_o)$-prearrangement. Obviously we have:

$$x(t,s^*) = s \quad \text{for } t \geq t_o + m_1.$$

It is possible that x (t,s^*) is a play of length K. This may happen for $t-t_o < m_K$. Later we shall also define $x(t, u^*)$ for every path u^* and for every period $t = t_o, t_o+1,\ldots$ There, too, $x(t, u^*)$ will describe what is predetermined for period t, once u^* has been played in the delay supergame.

The notation $x(\cdot, s^*)$ will be used for the initial status assignment. In the following $x(\cdot, s^*)$ will always be a fixed but arbitrary initial status assignment fitting the delay vector m and the stopping rule w.

4.4 Initial payoff vector

As in an ordinary supergame in a delay supergame payoffs for the periods played are accumulated as the game goes on. However, it will be convenient to permit the possibility that some fixed payoffs will be earned in addition to this. One may think of these payoffs as carried over from the past in a similar fashion as the initial status assignment $x(\cdot,s^*)$. Such payoffs carried over from the past arise naturally in subgames of delay supergames.

An *initial payoff vector*

$$c = (c_1,..., c_n)$$

is an n-vector with real valued components. c, is called *player i's initial payoff:* In the following c will always be a fixed but arbitrary initial payoff vector. The inclusion of an initial payoff vector among the specifications of a delay supergame has the purpose to define a delay supergame in such a way that the concept also covers the subgames of delay supergames.

4.5 The choice set function of the delay supergame

In 4.1 - 4.4 we have explained what has to be specified in order to describe a delay supergame associated to a bounded multistage game G: A delay vector m, an initial period t_o, a stopping rule w, an initial status assignment $x(\cdot,s^*)$, and an initial payoff vector c determine a delay supergame $G^* = \Gamma(G, m, w, x(\cdot,s^*), c)$. One may look at Γ as a function which assigns a multistage game G* to every bounded multistage game G augmented by the additional specifications shown as arguments of Γ. This will be made precise below.

The upper index * will be used wherever notation aims at details connected to G* which have a counterpart in G, e.g. preplays, plays, etc. . The star will not be used for symbols like m, w, x, and c which need not be distinguished from corresponding objects in G.

The start s* is a symbol which represents the situation before period t. We now recursively define the choice set function A*, the path set U*, and the status function $x(\cdot,\cdot)$ which assigns a path of G to every pair (t, u*) of a reachable period t and a path $u^* \in U^*$. It is clear that s* is a path and that for all reachable t the status x(t, s*) is already given by the initial status assignment. The recursive definition rests on this basis.

For i = 0,...,n and for every path u* of G* let $D_i(u^*)$ be the set of all reachable periods of the form $t_o+|u^*|+m_k$ with k = 1,..., K for which $A_i(x(t,u^*))$ is non-empty. $D_i(u^*)$ is interpreted as the set of all periods for which i has to make a decision at u*, if u* is a preplay. We refer to the elements of Di(u*) as the *aim periods* and to Di(u*) as the *aim period* set at u*. The union of all

$D_i(u^*)$ with i = 0,...,n is denoted by $D(u^*)$. This set is called the *joint aim period set* at u*.

For i = 0,...n and every path u* let $A^*\cdot_i(u^*)$ be the set of all systems of the form

$$a_i^* = (a_i^t)_{t \in D_i(u^*)} \text{ with } a_i^t \in A_i(x(t, u^*)) \text{ for all } t \in D_i(u^*)$$

If u* is a preplay, then the elements a_i^* of $A\cdot_i^*$ are *choices* of player i and in the case of a personal player i, all choices of player i. The component $a_i t$ of a_i^* is called player i's *decision* for t *at* u *. The random player has an additional choice ω, the *stopping choice,* at every preplay. If $A\cdot_o(u^*)$ is empty the random player also has a *continuation choice* $\tilde{\omega}$

$$A_o^*(u^*) = \omega \cup A\cdot_o(u^*) \text{ if } A\cdot_o(u^*) \neq \emptyset$$
$$A_o^*(u^*) = \{\omega,\tilde{\omega}\} \text{ if } A\cdot_o(u^*) = \emptyset$$

for every preplay u*. The *choice sets* of the personal players are

$$A_i^*(u^*) = A\cdot_i(u^*) \text{ for } i = 1,...,n$$

for every preplay u*. For a play z* we have:

$$A_i(z^*) = \emptyset \text{ for } i = 0,...,n$$

It still needs to be explained what distinguishes a play z* from a preplay u*. A play z* must be of the form

$$z^* = u^*a^* \text{ with } a^* \in A^*(u^*)$$

where u* is a preplay. A path z* of this form is a *play* if and only if one of the following conditions is satisfied.

(1) a* has the random component $a_o^* = \omega$

(2) a* does not have the random component $a_o^* = \omega$, the game G* is bounded and $t_o + |z^*| = T+1$ is unreachable

In case (2) we speak of a *maximal* play and in case (1) of a *submaximal* play. In unbounded delay supergames plays cannot end in any other way than by a random choice $a_o^* = \omega$. However, in a bounded delay supergame a play can extend over all $T-t_o+1$ reachable periods $t_o,...,T$. The length of a maximal play is $|z^*| = T-t_o+1$.

The periods of a play z* in which the random choice was not ω are called *unstopped.* The last period $t_o + |z^*| - 1$ of a submaximal play is called *stopped.* The set of all unstopped periods of a play z* is denoted by L(z*). The set L(z*) can be empty. This happens if already t_o is stopped. In 4.7 it will be explained that in the course of a play payoffs are accumulated for unstopped periods only.

In order to complete the definition of the choice set function we still have to describe how the status of a reachable period t changes in the course of playing the game. For every path of the form u*a* the *status* x(t,u*a*) of t at u*a* is recursively defined as follows:

$$x(t,u^*a^*) = x(t,u^*)a^t \text{ with } a^t \text{ in } a^* \text{ for } t \in D(u^*)$$

$$x(t,u^*a^*) = x(t,u^*) \text{ for } t \notin D(u^*)$$

for every preplay u* and every $a^* \in A^*(u^*)$. This means that the status of t at u* is changed by the decisions for t in a* but, of course, only if t is an aim period of u*. It is clear how the choice set function A*, the preplay set U*, the preplay set Z*, and the *status function* $x(\cdot,\cdot)$ are determined by the joint recursive definition given above.

4.6 The probability assignment of the delay supergame

The probability of a random choice a_o^* at a preplay u* according to the probability assignment p* of G* may be thought of as the result of independent random draws according to p for all aim periods in $D_o(u^*)$ combined with an independent decision on whether to stop with probability $w_{|u^*|}$ or to continue. In order to make this more precise consider a random choice a,* at a preplay u* of G*. For every $t \in D_o(u^*)$ let π^t be the probability

$$\pi^t = p_u(a_o^t) \quad \text{with } u = x(t,u^*) \text{ and } a_o^t \text{ in } a_o^*$$

Moreover let π be the product of all π^t with $t \in D_o(u^*)$. Then we have:

$$p_u^*(\omega) = w_{|u^*|}$$
$$p_u^*(\varpi) = 1-w_{|u^*|} \text{ for } A_{\cdot o}(u^*) = \emptyset$$
$$p_u^*(a_o^*) = (1-w_{|u^*|})\pi \text{ for } a_o^* \in A_{\cdot o}(u^*)$$

It can be seen without difficulty that the random stopping requirement imposed on w in 4.2 secures the random stopping condition of 2.2 for p*.

4.7 The payoff function of the delay supergame

A path u* of G* has the form of a sequence starting with s* and continuing with |u*| choice combinations at the periods $t_o,\dots,t_o+|u^*|-1$. Accordingly we say that periods t with $t_o \le t < t_o+|u^*|$ are *in the past* of u*. The other reachable periods from $t_o+|u^*|$ on are *in the future* of u*. All decisions for a period t in the past of u* which may have to be made as long as the status of t is a preplay, must be made before t or at t. Therefore the status x(t,u*) of a period in the past of u* must be a play of G.

In the delay supergame G* payoffs for a play z* are composed of initial payoffs and of payoffs for unstopped periods accumulated as z* is played:

$$h^*(z^*) = c + \sum_{t \in L(z^*)} h(x(t,z^*))$$

No payoffs are obtained for a *stopped* period, in which the random choice was ω. The choice of ω is thought of as immediately effective in the sense that the period is stopped already at its beginning.

From what has been explained in 4.5 and 4.6 it is clear that s*, A*, and p* satisfy the conditions jointly imposed on the start, the choice set function and the probability assignment of a multistage game. In order to see that G*= (s*,A*,p*,h*) has all the properties of a multistage game it remains to show that h* satisfies the boundedness condition of 2.3.

Since G is bounded and K is the maximum length of a play z of G it follows by the boundedness condition for G that we have

$$|h_i(z)| \leq C_o + KC_1$$

for some constants C_o and C_1. Let C_1^* be the right hand side of this inequality and let C_o^* be the maximum of the $|c_i|$. Obviously h* satisfies the boundedness condition with C_o^* and $C_1{}^*$ in the place of C_o and C_1.

We have now completed the definition of the delay supergame

$$G^* = (s^*,A^*,p^*,h^*) = \Gamma(G,m,w,x(\cdot,s^*),c)$$

and we have shown that G* is a multistage game.

4.8 Expected payoffs in delay supergames

The expected payoff vector H*(b*) for a strategy combination b* = $(b_1^*,...,b_n^*)$ of G* is defined in the same way as for multistage games in general. However, it will be useful to express H*(b*) in a way which focuses on decisions for periods rather than on choices at periods.

For every reachable period t let V_t^* be the set of all paths v* with $|v^*| = t-t_o+1$. Obviously a path $v^* \in V_t^*$ ends with a choice combination at period t. After period t all decisions for t have been made. Therefore the status $x(t,v^*)$ for $v^* \in V_t^*$ must be a play $z \in Z$. For every $z \in Z$ let $Vt^*(z)$ be the set of all $v^* \in V_t^*$ with $x(t,v^*) = z$. For every strategy combination $b^* = (b_1^*,...,b_n^*)$ and every reachable period $t \in R$ define

$$b^*(t,z) = \sum_{v^* \in V_t^*(z)} b^*(v^*)$$

and

$$F^t(b^*) = \sum_{z \in Z} b^*(t,z)h(z)$$